Generalizations of Totally Imprimitive Permutation Groups

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INTRODUCTION

In 1975, P. M. Neumann defined the totally imprimitive group for transitive finitary permutation groups in his article "The lawlessness of Groups of Finitary Permutations"[1]. Our aim is to extend to subgroups of symmetric groups the results obtained for totally imprimitive subgroups of finitary symmetric groups. We consider a generalization of totally imprimitive finitary permutation groups to a more general context, defining a property that we shall call the support-block property. Therefore, if G is a subgroup of $\mathrm{Sym}(\Omega)$, we can generalize for G some results that were obtained for totally imprimitive finitary permutation groups.

First, I will introduce the basic definitions necessary for our study, which are drawn from Permutation Groups by John D. Dixon and Brian Mortimer [2].

Definition
Let Ω be a nonempty set. A bijection of Ω onto itself is called a permutation of Ω. The set of all permutations of Ω forms a group, under composition of mappings, called the symmetric group on Ω. This group is denoted by $Sym(\Omega)$.

Definition
Let G be a group and Ω be a nonempty set. A group action is a function

$$\Omega \times G \to \Omega$$

$$(\alpha, x) \mapsto \alpha^x$$

satisfying the following two properties:
i) $\alpha^1 = \alpha$ for all $\alpha \in \Omega$, where 1 is the identity element of G.
ii) $\alpha^{xy} = (\alpha^x)^y$ for all $x, y \in G$.
Then we say that G acts on Ω.

Definition
Let G be a group acting on a set Ω. Then the set

$$\alpha^G = \{\alpha^x | x \in G\}$$

is called the orbit of α under G.

Definition
If $g \in G$ then the support of g is the set of elements in Ω moved by g:

$$supp(g) = \{\alpha \in \Omega | \alpha^g \neq \alpha\}.$$

Theorem
Let $g, h \in Sym(\Omega)$. Then

1. $supp(gh) \subseteq supp(g) \cup supp(h)$,
2. $supp(g^{-1}) = supp(g)$,
3. $supp(g) = \emptyset$ *if and only if* $g = 1$.

Definition
If $FSym(\Omega)$ is finitary symmetric group on Ω, then $FSym(\Omega)$ be the set of elements in $Sym(\Omega)$ which have finite support.

$$FSym(\Omega) = \{x \in S^{\Omega} : |supp(x)| < \infty\}$$

Definition
A group G acting on a set Ω is said to be transitive on Ω if for every $\alpha, \beta \in \Omega$ there exists $x \in G$ such that $\alpha^x = \beta$. Then it has only one orbit, and so $\alpha^G = \Omega$ for every $\alpha \in \Omega$

Definition
A group G acting on a set Ω, a nonempty subset Δ of Ω is called a block for G if for each $x \in G$ either $\Delta^x = \Delta$ or $\Delta^x \cap \Delta = \emptyset$.

Every group acting transitively on Ω has Ω and the singletons $\{\alpha\}$ ($\alpha \in \Omega$) as blocks; these are called the trivial blocks. Any other block is called nontrivial.

The group is primitive if G has no nontrivial blocks on Ω; otherwise G is called imprimitive.

If G has no maximal proper block and so there exists an infinite strictly ascending sequence of finite blocks

$$\Delta_1 \subset \Delta_2 \subset \dots$$

then G is totally imprimitive.

WREATH PRODUCT

In this section, the definition of the general wreath product given by P. Hall in his paper Wreath Powers and Characteristically Simple Groups is presented [3]. Let $\{H_\lambda : \lambda \in \Lambda\}$ be a family of transitive permutation groups, and let X_λ be the set on which H_λ acts. For each λ, let one element of X_λ be singled out and denoted by 1_λ.

Let X be the restricted product of the sets X_λ.[1] The elements of X are all the vectors of the form

$$x = (x_\lambda)_{\lambda \in \Lambda}$$

such that $x_\lambda \in X_\lambda$ for all λ and $x_\lambda = 1_\lambda$ except for finitely many values of λ.

[1] $X = Dr_{\lambda \in \Lambda} X_\lambda$

Now let Λ be ordered: this implies that, for any given λ and μ in Λ, exactly one of the relations $\lambda < \mu$, $\lambda = \mu$ or $\lambda > \mu$ holds.[2] If $x, y \in X$, and $\mu, \lambda \in \Lambda$, then

$$x \equiv y \pmod{\lambda}$$

to mean that $x_\mu = y_\mu$ for all $\mu > \lambda$. In particular,

$$x \equiv 1 \pmod{\lambda}$$

means that $x_\mu = 1_\mu$ for all $\mu > \lambda$.

[2]A relation that has the properties of reflection, inverse symmetry, and transitivity on a set is called a partial ordering relation, and this set is called a partially ordered set. If any two elements of a partially ordered set are related by the partial ordering relation, this relation is called a linear ordering relation, and the set is called a ordered set.

Let $\xi \in H_\lambda$ and $x \in X$. Let ξ_x be the permutation of X. Then only elements congruent to 1 modulo λ are affected by ξ_x, which are the only λ-th coordinates. If

$$x \equiv 1 \pmod{\lambda},$$

then $(x\xi)_\lambda = x_\lambda \xi$ and $(x\xi)_\mu = x_\mu$ for all

$$\mu \neq \lambda.$$

If

$$x \not\equiv 1 \pmod{\lambda},$$

then $x\xi_x = x$.

Since

$$\begin{array}{rccc} \rho: & H_\lambda & \to & Sym(X) \\ & \xi & \mapsto & \xi_x \end{array}$$

is homomorphism, H_λ is represented as a permutation group of X.

Let J_λ be the group formed by all ξ_x permutations.

$$J_\lambda = \{\xi_x \in Sym(X) | \xi \in H_\lambda\} \cong H_\lambda.$$

The wreath product W of H_λ groups is a group of permutations of X generated by J_λ groups:

$$\begin{aligned} W &= Wr_{\lambda \in \Lambda} H_\lambda \\ &= \langle J_\lambda | \lambda \in \Lambda \rangle. \end{aligned}$$

Theorem
W is transitive on X.

Proof.
To prove this, it is enough to show that there is an element $\xi \in W$ such that $1\xi = x$ for all $x \in X$.

For finitely many $\lambda_k \in \Lambda$ except $x_{\lambda_k} = 1_{\lambda_k}$, so let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \Lambda$ such that $x_{\lambda_i} \neq 1_{\lambda_i}$. Also, since Λ is linearly ordered, we can assume that $\lambda_1 < \lambda_2 < \ldots < \lambda_r$. Since groups H_{λ_i} act transitively on X_{λ_i}, there exists $\xi_i \in H_{\lambda_i}$ such that $1_{\lambda_i}\xi_i = x_{\lambda_i}$. Since the elements $(\xi_1)_x, (\xi_2)_x, \ldots, (\xi_r)_x$ are permutations of X, $(\xi_1)_x(\xi_2)_x \ldots (\xi_r)_x \in \langle J_{\lambda_i} | i = 1, 2, \ldots, r \rangle$, i.e. $(\xi_1)_x(\xi_2)_x \ldots (\xi_r)_x \in W$. Let $\xi = (\xi_1)_x(\xi_2)_x \ldots (\xi_r)_x$. Thus

$$\begin{aligned}
1\xi &= 1((\xi_1)_x, (\xi_2)_x, \ldots, (\xi_r)_x) \\
&= (1_{\lambda_1}, 1_{\lambda_2}, \ldots, 1_{\lambda_r}, 1_{\lambda_i}, \ldots)((\xi_1)_x, (\xi_2)_x, \ldots, (\xi_r)_x) \\
&= (x_{\lambda_1}, x_{\lambda_2}, \ldots, x_{\lambda_r}, x_{\lambda_{r+1}}, \ldots) \\
&= (x_\lambda)_{\lambda \in \Lambda} \\
&= x
\end{aligned}$$

As a result, it is proven that W affects X transitively. Here, since the change of the constant element 1 does not affect W, for each $u \in X$ there exists $\xi \in W$ such that $u\xi = x$. $\square$

SUPPORT-BLOCK PROPERTY

Let G be a transitive subgroup of $\mathrm{Sym}(\Omega)$. If for every finitely generated subgroup F of G, there is a proper G-block Δ (i.e. $\Delta \neq \Omega$) such that

$$\mathsf{supp}(F) \subseteq \Delta,$$

then G satisfies the *support − block property* ($\mathcal{SBP}$ in short).

In Section 4 of the paper "Finitary Representations and Images of Transitive Finitary Permutation Groups" by F. Leinen and O. Puglisi, an example is given that the infinitely iterated wreath product of finite transitive permutation groups is a totally imprimitive finitary permutation group [4]. We study this example by considering the infinitely iterated general wreath product of infinitely transitive permutation groups, and we show that it satisfies the support block property.

Example

Let W be the infinite iterated wreath product defined as

$$W = H_0 wr_{\Omega_1} H_1 wr_{\Omega_2} H_2 wr_{\Omega_3} \dots wr_{\Omega_i} H_i \dots$$

where for each $n \in \omega$, $H_n \leq \mathrm{Sym}(\Omega_n)$ is a transitive permutation group. We can also consider W as the general wreath product $wr_{i\in\omega} H_i$ (see [3]) on the set

$$\Omega = \{(\alpha_n)_{n\in\omega} \in Dr_{n\in\omega}\Omega_n | \alpha_n = 1_n \text{ for all but finitely many } n \in \omega\}$$

via

$$(\alpha_n)_{n\in\omega} x = \begin{cases} \alpha_n x & \text{if } n = i \text{ and } \alpha_m = 1_m \text{ for all } m > n \\ \alpha_n & \textit{else} \end{cases}$$

for $x \in H_i$.

Such a wreath product W is the direct limit of the iterated wreath products $W_0 = H_0$ and

$$W_n = W_{n-1} wr_{\Omega_n} H_n = H_0 wr_{\Omega_1} H_1 wr_{\Omega_2} H_2 wr_{\Omega_3} \dots wr_{\Omega_n} H_n$$

for $n \geq 1$; respect to the canonical embedding of W_{n-1} onto a fixed component of the base group of W_n (corresponding to $1_n \in \Omega_n$).

Let $\Delta_n = \mathrm{supp}_\Omega W_n = \{(\alpha_n)_{n\in\omega} \in \Omega | \alpha_m = 1_m$ for all $m > n\}$. If $x \in W_n$, then $\Delta_n^x = \Delta_n$. If $x \notin W_n$, then $x \in W_m$ and there is an $m \in \mathbb{Z}^+$ such that $m > n$. If $1_m \in \mathrm{supp}(x)$, then $\Delta_n^x \cap \Delta_n = \emptyset$, If $1_m \notin \mathrm{supp}(x)$, then $\Delta_n^x = \Delta_n$. Therefore Δ_n is a W-block in Ω for each n. Since W_n affects n-th coordinate of Ω, only $\Delta_n \neq \Omega$.

Let F be a finitely generated subgroup of W, then $F \leq W_n$ for some $n \in \omega$. Hence

$$\text{supp}_\Omega F \subseteq \Delta_n = \text{supp}_\Omega W_n.$$

Consequently, for every finitely generated subgroup F of W, there is a W-block Δ_n for some $n \in \omega$ such that

$$\text{supp}_\Omega F \subseteq \Delta_n = \text{supp}_\Omega W_n.$$

i.e. W satisfies $\mathcal{SBP}$. Since W is the direct limit of W_n groups,

$$W_1 < W_2 < \dots$$

and so

$$supp(W_1) \subset supp(W_2) \subset \dots$$

There exists an infinite strictly ascending sequence of W-blocks

$$\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \dots$$

Suppose that $(x_n)_{n\in\omega} \in \Omega$, then there is $s \in \mathbb{Z}^+$ such that $x_r = 1_r$ for $r > s$ by the definition of Ω, so $(x_n)_{n\in\omega} \in \Delta_s$. Hence $\Omega = \bigcup_{n\in\omega} \Delta_n$.

In the following, I will present some of the results we have obtained. In his paper *"The Structure of Finitary Permutation Groups"*, P. M. Neumann studied Lemma 3.2, Theorem 3.3, Corollary 3.4, and Theorem 1 in the context of totally imprimitive groups. We revisit these results for groups that satisfy the support-block property (SBP). The proofs are similar to those given by Neumann.

Lemma

Let G be a transitive permutation group with $\mathcal{SBP}$ and N be a transitive normal subgroup of G on an infinite set Ω. If $g \in G$ and Δ is a nontrivial proper G-block such that supp$(g) \subseteq \Delta$ then there exists $h \in N$ such that $\alpha^g = \alpha^h$ for all $\alpha \in \Delta$.

Proof.
Since N is a transitive subgroup of G on Ω, there is $x \in N$ such that $\Delta \cap \Delta^x = \emptyset$. Now $h := [x, g] \in N$ and since $\mathrm{supp}((g^{-1})) \subseteq \Delta$ and

$$\mathrm{supp}((g^{-1})^x) = (\mathrm{supp}(g^{-1}))^x \subseteq \Delta^x,$$

we have that $\alpha^g = \alpha^h = (\alpha^{(g^{-1})^x})^g$ for all $\alpha \in \Delta$.

□

Theorem
Let G be a transitive permutation group with $\mathcal{SBP}$ and N be a transitive normal subgroup of G on an infinite set Ω. If M is a normal subgroup of N, then M is normal in G.

Proof.
If $m \in M$ and $g \in G$, then there is a nontrivial proper G-block Δ containing supp($\langle m, g\rangle$). By Lemma, there exists $h \in N$ such that $\alpha^g = \alpha^h$ for all $\alpha \in \Delta$. Since $\mathrm{supp}(m) \subseteq \Delta$ and $\mathrm{supp}(g^x) \subseteq \Delta^x$, $\mathrm{supp}(m) \cap \mathrm{supp}(g^x) = \emptyset$. Then $m^h = m^{[x,g]} = m^g$, so $m^g \in M$ and hence M is normal in G.

$\square$

Corollary
Let G be a transitive permutation group with $\mathcal{SBP}$. Then transitive subnormal subgroups are normal in G.

Theorem

Let G be a transitive permutation group on an infinite set Ω with $\mathcal{SBP}$. If N is a transitive normal subgroup of G, then the derived subgroup of G is a subgroup of N ($G' \leq N$).

Proof.
Suppose that $g, h \in G$. Then there is a nontrivial proper G-block Δ such that $\mathrm{supp}(\langle g, h\rangle) \subseteq \Delta$. Since N is transitive on Ω, there is $x \in N$ so that $\Delta \cap \Delta^x = \emptyset$. We can choose $c := g^{-1}x^{-1}gxh^{-1}x^{-1}hx(gh)x^{-1}(gh)^{-1}x$. Since

$$\mathrm{supp}(\langle x^{-1}gx\rangle), \mathrm{supp}(\langle x^{-1}hx\rangle), \mathrm{supp}(\langle x^{-1}(gh)^{-1}x\rangle) \subseteq \Delta^x,$$

these subgroups commute with all of g, h and gh. Therefore, c can be rewritten as

$$c := g^{-1}h^{-1}ghx^{-1}gxx^{-1}hxx^{-1}(gh)^{-1}x.$$

Thus c is just the commutator $[g, h]$. On the other hand, c is a product of conjugates of x and x^{-1}, and so $c \in N$. Hence all commutators lie in N and $G' \leq N$. $\square$

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[5] P.M. Neumann, The structure of finitary permutation groups, Arch. Math. 27 (1976) 3-17.

THANKS...