Set-theoretic solutions of the Yang–Baxter equation associated with g-digroups

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This talk is based on some results obtained in:

A. Albano, P. Stefanelli, *Generalized digroups, di-skew braces and solutions of the set-theoretic Yang-Baxter equation*, arXiv:2505.15387.

We will

- review basic facts on self-distributivity in the YBE;
- determine how generalized digroups yield solutions;
- compare the latter with skew brace solutions.

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The set-theoretic Yang–Baxter equation

V. G. Drinfel'd, *On some unsolved problems in quantum group theory*, in Quantum groups, (Springer) Lecture Notes in Math. 1510 (1990), 1-8.

If D is a set, a map $r : D \times D \to D \times D$ is called a *set-theoretic solution* to the *YBE* if it satisfies the *braid relation*:

$$(r \times \mathrm{id}_D)(\mathrm{id}_D \times r)(r \times \mathrm{id}_D) = (\mathrm{id}_D \times r)(r \times \mathrm{id}_D)(\mathrm{id}_D \times r)$$

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If we project r onto its components and write

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

where $\lambda_a, \rho_b : D \to D$ are maps, for all $a, b \in D$, then r is named:

- *left non-degenerate* if λ_a is bijective, for all $a \in D$;
- *right non-degenerate* if ρ_b is bijective, for all $b \in D$;
- *non-degenerate* if it is both left and right non-degenerate.

How to tell solutions apart

Let (D, r) and (X, s) be solutions. Then, they are called

- ***D*-isomorphic** if there exists a bijection $F : D \times D \to X \times X$ such that

$$Fr = sF\,.$$

- **equivalent** if there exists a bijection $f : D \to X$ such that

$$(f \times f)r = s(f \times f)\,.$$

Self-distributive structures

A *shelf* $(D, \triangleright)$ is a set D equipped with a binary operation $\triangleright$ such that:

$$\forall x, y, z \in D \quad x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)\,.$$

A shelf $(X, \triangleright)$ is called a

- *rack*, if the map $L_x : X \ni y \mapsto x \triangleright y \in X$ is bijective, for all $x \in D$;
- *quandle*, if $(X, \triangleright)$ is a rack such that $x \triangleright x = x$, for all $x \in D$.

D. Joyce, *A classifying invariant of knots, the knot quandle*, J. Pure Appl. Algebra 23 (1982), 37-65.

S. V. Matveev, *Distributive groupoids in knot theory*, Mat. Sb. (N.S.) 119 (1982), 78-88.

Self-distributivity in the YBE

A *twist* of a shelf $(D, \triangleright)$ is a map $\lambda : D \to \mathrm{Aut}(D, \triangleright)$ such that

$$\forall x, y \in D \quad \lambda_x \lambda_y = \lambda_{\lambda_x(y)} \lambda_{\lambda^{-1}_{\lambda_x(y)}(\lambda_x(y) \triangleright x)}$$

Theorem [Doikou, Rybołowicz, Stefanelli (2024)]

If $(D, \triangleright)$ is a shelf and $\lambda : D \to \mathrm{Sym}(D)$ a map, then

$$r_\lambda : D \times D \to D \times D\,,\ r(x, y) = (\lambda_x(y), \lambda^{-1}_{\lambda_x(y)}(\lambda_x(y) \triangleright x))\,,$$

is a left non-degenerate solution if and only λ is a twist of $(D, \triangleright)$.
Moreover, all *left non-degenerate solutions* can be constructed in this way.

Conjugation quandle: let G be a group and set

$$\forall x, y \in G \quad x \triangleright y := x^{-1}yx$$

Then, $(G, \triangleright)$ is a quandle which will be denoted as $\mathrm{Conj}(G)$.

Conjugation rack: let D be a generalized digroup and set

$$\forall x, y \in D \quad x \triangleright y := x^{-1} \vdash y \dashv x$$

Then, $(D, \triangleright)$ is a rack which will be denoted as $\mathrm{Conj}(D, \vdash, \dashv)$.

Disemigroups

J.-L. Loday, *Dialgebras*, in Dialgebras and related operads, Lecture Notes in Math. 1763 (2001), 7-66.

A *disemigroup* $(D, \vdash, \dashv)$ is the datum of two binary *associative* operations $\vdash, \dashv$ on a set D satisfying the following properties, for all $x, y, z \in D$:

$$\begin{aligned} x \vdash (y \dashv z) &= (x \vdash y) \dashv z\,, && \text{(inner associativity)} \\ x \dashv (y \vdash z) &= x \dashv (y \dashv z)\,, && \text{(right bar-side irrelevance)} \\ (x \vdash y) \vdash z &= (x \dashv y) \vdash z\,, && \text{(left bar-side irrelevance)} \end{aligned}$$

A *bar-unit* is an element $e \in D$ with the following property

$$\forall x \in D \quad e \vdash x = x = x \dashv e$$

The set of all bar-units is called the *halo* of D and denoted as $E(D, \vdash, \dashv)$.

Generalized digroups

M. K. Kinyon, *Leibniz algebras, Lie racks, and digroups*, J. Lie Theory 17 (2007), 99-114.

O. P. Salazar-Díaz, R. E. Velásquez Ossa, L. A. Wills Toro, *Generalized digroups*, Comm. Algebra 44 (2016), 2760-2785.

A *generalized digroup* (g-digroup) is a disemigroup $(D, \vdash, \dashv)$ such that

i) there exists a bar-unit, i.e. $E(D, \vdash, \dashv) \neq \emptyset$;

ii) for all $e \in E(D, \vdash, \dashv)$ and $x \in D$ there exists a unique pair of elements $x^{I_e}, x^{J_e} \in D$ such that

$$x^{I_e} \dashv x = e = x \vdash x^{J_e}$$

The structure of g-digroups

If $(D, \cdot)$ is a group then $(D, \cdot, \cdot)$ trivially is a g-digroup.

Let G be a group endowed with a right action $\psi : G \to \mathrm{Sym}(E)$ on a set E. If on $D := G \times E$, for all $g, h \in G$ and $e, f \in E$, we define

$$(g, e) \vdash (h, f) := (gh, f),$$
$$(g, e) \dashv (h, f) := (gh, \psi_h(e)),$$

then $(D, \vdash, \dashv)$ is a g-digroup which we denote by $G \dashv\vdash_{\psi} E$. Note that it satisfies $E(D, \vdash, \dashv) = \{1_G\} \times E$.

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Theorem [Kinyon (2007) and Salazar-Díaz et al. (2016)]

If $(D, \vdash, \dashv)$ is a g-digroup then there exists a group G and a right action $\psi : G \to E(D, \vdash, \dashv)$ such that

$$D \cong G \dashv\vdash_{\psi} E(D, \vdash, \dashv).$$

The conjugation rack

There is a well-defined binary operation $\triangleright$ on D given by

$$x \triangleright y := x^{-1} \vdash y \dashv x \,.$$

The pair $(D, \triangleright)$ is a rack, called the *conjugation rack* of $(D, \vdash, \dashv)$ and denoted by $\mathrm{Conj}(D, \vdash, \dashv)$.

In particular, if $D = G \dashv\vdash_{\psi} E$ then for all $g, h \in D$ and $e, f \in E$ we have

$$(g, e) \triangleright (h, f) = \left(g^{-1}hg, \psi_g(f)\right) \,.$$

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$$(g, e) \triangleright (h, f) = \left(g^{-1}hg, \psi_g(f)\right) \,.$$

$(D, \triangleright)$ is a quandle $\iff \forall x \in D, e \in E(D, \vdash, \dashv) \quad x \triangleright e = e$

Di-skew braces

A. Albano, P. Stefanelli, *Generalized digroups, di-skew braces and solutions of the set-theoretic Yang-Baxter equation*, arXiv:2505.15387.

Definition

A *di-skew brace* $(D, \vdash, \dashv, \circ)$ is the datum of a g-digroup $(D, \vdash, \dashv)$ and a *right group* $(D, \circ)$ such that the following hold, for all $x, y, z \in D$

$$x \circ (y \vdash z) = x \circ y \vdash x^{-1} \vdash x \circ z\,,$$
$$x \circ (y \dashv z) = x \circ y \dashv x^{-1} \dashv x \circ z\,,$$
$$(x \vdash y) \circ z = (x \dashv y) \circ z\,.$$

Examples:

- **trivial di-skew brace** $\longrightarrow (D, \vdash, \dashv, \vdash)$;
- **almost-trivial di-skew brace** $\longrightarrow (D, \vdash, \dashv, \dashv^{\mathrm{op}})$;
- **skew braces** $\longrightarrow (D, +, +, \circ)$.

Di-skew braces and solutions

Let $(D, \vdash, \dashv, \circ)$ be a di-skew brace. For all $x \in D$, one can define a digroup automorphism $\lambda_x \in \mathrm{Aut}(D, \vdash, \dashv)$ as follows

$$\lambda_x : D \to D\,,\ \lambda_x(y) = x^{-1} \vdash x \circ y$$

The map $\lambda : (D, \circ) \to \mathrm{Aut}(D, \vdash, \dashv)$ is a digroup homomorphism.

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Theorem

Let $(D, \vdash, \dashv, \circ)$ be a di-skew brace and consider $(D, \triangleright) = \mathrm{Conj}(D, \vdash, \dashv)$. Then, the map $\lambda : D \to \mathrm{Aut}(D, \triangleright)$ is a twist. In particular, the map $r : D \times D \to D \times D$ defined by

$$r(x, y) = \Big(\lambda_x(y), (\lambda_x(y))^- \circ (x \dashv \lambda_x(y))\Big)$$

is a bijective non-degenerate solution.

Order will tell us apart

Let $(D, \vdash, \dashv, \circ)$ be a di-skew brace and let r be its associated solution.

Proposition

In the above notation, if we let $(D, \vdash, \dashv) \cong G \dashv\vdash_\psi E$ then

$$o(r) = 2 \cdot \operatorname{lcm}\left(\exp\left(G/Z(G)\right), N_\psi\right) ,$$

where $N_\psi = \inf\left\{n \in \mathbb{N} \mid \forall x, y \in G \quad \left(\psi_x \psi_y\right)^n \psi_x^{-n} = \mathrm{id}_E\right\}$

This extends Theorem 4.13 in [Smoktunowicz, Vendramin (2018)].

An example

Let $E = \{1, 2, 3\}$ and consider $G = \langle (12) \rangle$ together with its natural action $\psi : G \to \mathrm{Sym}(E)$. Then, any di-skew brace $(D, \vdash, \dashv, \circ)$ with $(D, \vdash, \dashv) \cong G \dashv\vdash_\psi E$ determines a solution r_D having

$$o(r_D) = 4\,.$$

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$$o(r_D) = 4\,.$$

All skew braces $(B, +, \circ)$ on a set of $|D| = 6$ elements satisfy

$$o(r_B) \in \{2, 12\}$$

depending on whether $(B, +) \cong C_6$ or $(B, +) \cong \mathrm{Sym}_3$, respectively.

The solutions r_D cannot be D-isomorphic to a skew brace solution.

Thank you for your attention!

Grazie per l'attenzione!

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