Near braces, \( p \)-deformed braided groups & quasi-bialgebras

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YBE introduced: Yang, study of $N$ particle in $\delta$ potential & Baxter, study of XYZ model.

YBE and the $R$-matrix are the quantum analogues of the classical YBE and the classical $r$-matrix (Sklyanin bracket) [Semenof-Tjian-Shanski Sklyanin ...], classical intergrable systems.

Fundamental equ. in QISM formulation [Faddeev, Tahktajan, Kulish, Sklyanin, Reshetikhin...] & quantum algebras (deformed Lie algebras) [Drinfeld, Jimbo].

Solving YBE: e.g. using braids and Hecke algebras Baxterization [Jimbo], or using linear intertwining relations (quantum group symmetry) [Kulish].
[Drinfeld] introduced the “Set-theoretic YBE”.


Connections to: geometric crystals [Berenstein & Kazhdan, Etingof] and cellular automatons [Hatayama, Kuniba & Takagi].

Classical discrete integrable systems (YB maps), quad-graph, discrete maps: [Veselov, Bobenko, Suris, Papageorgiou, Tongas,...]

Set theoretical involutive solutions of YBE from braces: [Rump, Guarnieri, Vendramin, Gateva-Ivanova, Cedó, Jespers, Okniński, Smoktunowicz,...]
We recall basic definitions and results about braces and set-theoretic solutions of the Yang-Baxter equation. We introduce slightly generalized algebraic structures called near-braces associated to solutions of the YBE. Starting from set theoretic solutions with extra fixed parameters, we reconstruct the near brace \((\mathcal{A}, \mathcal{D}, \text{Rybolowicz})\) and vise versa. The notion of the p-deformed braided groups is also introduced leading to generalized solutions of the YBE.

We recall the definitions of the quasi bi-algebra and we then move on to study the quantum algebra associated to set-theoretic solutions of the YBE \((\text{Etingof, Shedler, Soloviev})\). The quantum algebra is a quasi bialgebra & the set-theoretic solutions are obtained via an admissible Drinfeld twist from a derived solution \((\mathcal{A}, \text{Ghionis, Vlaar}; \mathcal{A}, \text{Rybolowicz}; \mathcal{A})\).
Let a set $X = \{x_1, \ldots, x_N\}$ and $\tilde{\rho} : X \times X \to X \times X$. Denote

$$\tilde{\rho}(x, y) = (\sigma_x(y), \tau_y(x))$$

1. $(X, \tilde{\rho})$ non-degenerate: $\sigma_x$ and $\tau_y$ are bijective functions
2. $(X, \tilde{\rho})$ involutive: $\tilde{\rho}(\sigma_x(y), \tau_y(x)) = (x, y)$, $(\tilde{\rho} \tilde{\rho}(x, y) = (x, y))$

Suppose $(X, \tilde{\rho})$ is an involutive, non-degenerate set-theoretic solution of the Braid equation:

Braid equation

$$(\tilde{\rho} \times \text{Id}_X)(\text{Id}_X \times \tilde{\rho})(\tilde{\rho} \times \text{Id}_X) = (\text{Id}_X \times \tilde{\rho})(\tilde{\rho} \times \text{Id}_X)(\text{Id}_X \times \tilde{\rho}).$$
**Definition (Rump; Guarnieri & Vendramin)**

A *left skew brace* is a set $B$ together with two group operations $+ : B \times B \to B$, the first is called addition and the second is called multiplication, such that $\forall a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c.$$ 

If $+$ is an abelian group operation $B$ is called a *left brace*. Moreover, if $B$ is a left skew brace and $\forall a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then $B$ is called a *skew brace*. Analogously if $+$ is abelian and $B$ is a skew brace, then $B$ is called a *brace*.

- The additive identity of a left skew brace $B$ will be denoted by 0 and the multiplicative identity by 1. In every left skew brace $0 = 1$. 
Theorem (Rump)

Let \((B, +, \circ)\) be a brace and for \(x, y \in B\) define \(\check{r}(x, y) = (\sigma_x(y), \tau_y(x))\), the YB map of \((B, +, \circ)\):

\[
\sigma_x(y) = x \circ y - x, \quad \tau_y(x) = (\sigma_x(y))^{-1} \circ x - (\sigma_x(y))^{-1}.
\]

Then \((B, \check{r})\) is a non-degenerate, involutive solution of the set-theoretic YBE. Conversely, for every non-degenerate, involutive solution of the set-theoretic YBE \((X, \check{r})\) there is a brace \((B, +, \circ)\): \(X \subseteq B\) and \(\check{r}\) is the restriction of the YB map of \((B, +, \circ)\) to \(X \times X\).

See also [Cedo, Jespers, Okninski].

Guarnieri & Vendramin generalized Rump’s results for non-involutive, non-degenerate, set-theoretic solutions using skew braces.
A generalized version of the set-theoretic solution via some kind of “z-deformation”. Let \( z \in X \) be fixed, then we denote

\[
\tilde{r}_z(x, y) = (\sigma^z_x(y), \tau^z_y(x)).
\]

\( \tilde{r} \) is non-degenerate if \( \sigma^z_x \) and \( \tau^z_y \) are bijective maps.

By requiring \( (X, \tilde{r}_z) \) to be a solution of the braid equation:

\[
(\tilde{r} \times \text{id})(\text{id} \times \tilde{r})(\tilde{r} \times \text{id}) = (\text{id} \times \tilde{r})(\tilde{r} \times \text{id})(\text{id} \times \tilde{r}),
\]

we obtain the fundamental constraints:

\[
\begin{align*}
\sigma^z_{\eta}(\sigma^z_x(y)) & = \sigma^z_{\sigma^z_{\eta}(x)}(\sigma^z_{\tau^z_x(\eta)}(y)), \quad (1) \\
\tau^z_y(\tau^z_x(\eta)) & = \tau^z_{\tau^z_y(x)}(\tau^z_{\sigma^z_x(y)}(\eta)), \quad (2) \\
\tau^z_{\sigma^z_{\tau^z_x(\eta)}(y)}(\sigma^z_{\eta}(x)) & = \sigma^z_{\tau^z_{\sigma^z_x(y)}(\eta)}(\tau^z_y(x)). \quad (3)
\end{align*}
\]
Theorem

Let $B$ be a left skew brace and $z \in B$ such that $\forall a, b, c \in B$, 
$(a - b + c) \circ z = a \circ z - b \circ z + c \circ z$. Then we can define a map $\tilde{\tau}_z : B \times B \to B \times B$ given by

$$
\tilde{\tau}_z(a, b) = (\sigma^z_a(b), \tau^z_b(a)) := (a \circ b - a \circ z + z, (a \circ b - a \circ z + z)^{-1} \circ a \circ b),
$$

where $(a \circ b - a \circ z + z)^{-1}$ is the inverse in the group $(B, \circ)$. The pair $(B, \tilde{\tau})$ is a solution of the braid equation.

**Proof.** I show that the maps $\sigma^z, \tau^z$ satisfy the constraints (1)-(3) [AD & Rybolowicz].

1. **Remark.** We note that $\sigma^z_x, \tau^z_y$ as defined in the Theorem are bijections and thus $\tilde{\tau}$ is non-degenerate.

2. **Remark.** For $z = 1$, Guarnieri-Vedramin skew braces are recovered; if in addition $(B, +)$ is an abelian group Rump’s braces are recovered and $\tilde{\tau}_{z=1}$ becomes involutive($\sigma^z_{\sigma^z_x(y)}(\tau^z_y(x)) = x$ and $\tau^z_{\tau^z_y(x)}(\sigma^z_x(y)) = y$). For $z \neq 1$ the solutions are not involutive in general, even in the case of braces.

**σ** is a group action but $\tau$ is not!!
The inverse $\tilde{r}$ matrix

- The explicit expressions of the inverse $\tilde{r}_z$-matrices & the bijective maps:

**Proposition**

Let $\tilde{r}_z, \tilde{r}^*_z : X \times X$ be solutions of the braid equations, such that $\tilde{r}^*_z : (x,y) \mapsto (\hat{\sigma}_x^z(y), \hat{\tau}_y^z(x)), \tilde{r} : (x,y) \mapsto (\sigma_x^z(y), \tau_y^z(x))$.

1. $\tilde{r}^* = \tilde{r}^{-1}$ if and only if

$$
\hat{\sigma}_x^z(y)(\tau_y^z(x)) = x, \quad \hat{\tau}_y^z(x)(\sigma_x^z(y)) = y, \quad \sigma_{\hat{x}}^z(y)(\hat{\tau}_y^z(x)) = x, \quad \tau_{\hat{y}}^z(x)(\sigma_x^z(y)) = y.
$$

2. If $\sigma_x^z(y) = x \circ y - x \circ z + z$, $\tau_y^z(x) = \sigma_x^z(y)^{-1} \circ x \circ y$, then

$$
\hat{\sigma}_x^z(y) = -x \circ z^{-1} + x \circ y \circ z^{-1}, \quad \hat{\tau}_x^z(y) = \hat{\sigma}_x^z(y)^{-1} \circ x \circ y.
$$
**Example (1)**

Let us consider a set $\text{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z} \right\}$ together with two binary operations $(a, b) \mapsto^+ a - 1 + b$ and $(a, b) \mapsto^\circ a \cdot b$, where $+,$ $\cdot$ are addition and multiplication of rational numbers, respectively. The triple $(\text{Odd}, +, \circ)$ is a brace. The solution $\tilde{r}_z$ is involutive if and only if $a \cdot b - a \cdot z + z = a \cdot b - a + 1$ if and only if $(z - 1) \cdot (1 - a) = 0, \forall a, b \in B$. Therefore, $\forall z \neq 1, \tilde{r}_z$ is non-involutive. Moreover, $\tilde{r}_z = \tilde{r}_w$ if and only if $-a \cdot z + z = -a \cdot w + w$, that is if $z = w$.

**Example (2)**

Let us consider a ring $\mathbb{Z}/8\mathbb{Z}$. A triple

$$\left( \text{OM} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \{1, 3, 5, 7\}, \ b, c \in \{0, 2, 4, 6\} \right\}, +_I, \circ \right)$$

is a brace, where $(A, B) \mapsto^+_I A - I + B, (A, B) \mapsto^\circ A \cdot B$, and $+,$ $\cdot$ are addition and multiplication of two by two matrices over $\mathbb{Z}/8\mathbb{Z}$, respectively. Moreover one can easily check that two solutions $\tilde{r}_A$ and $\tilde{r}_B$ are equal if and only if $(D - I) \cdot (B - A) = 0 \pmod{8} \ \forall D \in \text{OM}$.

The examples above are inspired by works on trusses, paragons...: [Brzezinski], [Brzezinski & Rybolowicz].
Let $X$ be a set with a group operation $\circ : X \times X \to X$, with a neutral element $1 \in X$ and an inverse $x^{-1} \in X$, $\forall x \in X$. Let $\sigma^z_x : X \to X$, be a family of bijective functions: $y \mapsto \sigma^z_x(y)$, where $z \in X$ is some fixed parameter. We define another binary operation $+: X \times X \to X$, such that

$$y + x := x \circ \sigma^z_{x^{-1}}(y \circ z) \circ z^{-1}$$

For convenience we will omit henceforth the fixed $z \in X$ in $\sigma^z_x(y)$. We assume that $+$ is associative.
Focus on non-degenerate, invertible solutions $\check{r}$ and $\sigma_x$ and $\tau_y$ are bijections:

$$\sigma_x^{-1}(\sigma_x(y)) = \sigma_x(\sigma_x^{-1}(y)) = y, \quad \tau_y^{-1}(\tau_y(x)) = \tau_y(\tau_y^{-1}(x)) = x$$

Let $\check{r}^{-1}(x,y) = (\check{\sigma}_x(y), \check{\tau}_y(x))$ exist with $\check{\sigma}_x$, $\check{\tau}_y$ being also bijections:

$$\sigma_{\check{\sigma}_x(y)}(\check{\tau}_y(x)) = x = \check{\sigma}_{\check{\sigma}_x(y)}(\check{\tau}_y(x)), \quad \tau_{\check{\tau}_y(x)}(\check{\sigma}_x(y)) = y = \check{\tau}_{\check{\tau}_y(x)}(\check{\sigma}_x(y)).$$

$$\check{\sigma}^{-1}_{\sigma_x(y)}(x) = \tau_y(x), \quad \check{\tau}^{-1}_{\tau_y(x)}(y) = \sigma_x(y).$$

Assume that $\check{\sigma}$ has the general form

$$\check{\sigma}_x(y) := x \circ (x^{-1} \circ z_2 + y \circ z_1) \circ \xi,$$

the parameters $z_{1,2}$, $\xi$ to be identified.
Useful Lemmas:

**Lemma 1**

For all \( x \in X \), the operations \(+x\), \( x+ : X \to X\) are bijections.

**Lemma 2**

Let \((X, +)\) be a semigroup, then \( \forall x \in X, \exists 0_x \in X \), such that \( 0_x + x = x \). Moreover, \( \forall x, y \in X, 0_x = 0_y = 0 \), i.e. 0 is the unique left neutral element. The left neutral element 0 is also right neutral element.

**Lemma 3**

Let 0 be the neutral element in \((X, +)\), then \( \forall x \in X, \exists -x \in X \), such that \(-x + x = 0\) (left inverse). Moreover, \(-x \in X\) is a right inverse, i.e. \( x + (-x) = 0 \) \( \forall x \in X \). That is \((X, +, 0)\) is a group.
Let \((X, \circ)\) be a group and \(\tilde{r} : X \times X \to X \times X\) be such that \(\tilde{r}(x, y) = (\sigma_x(y), \tau_x(y))\) is a non-degenerate solution of the set-theoretic braid equation. Moreover, we assume that:

A. The pair \((X, +)\) (+ as defined) is a group.

B. There exists \(\phi : X \to X\) such that for all \(a, b, c \in X\)
\[a \circ (b + c) = a \circ b + \phi(a) + a \circ c.\]

C. For \(h \in \{z, \xi\} \in X\) appearing in \(\sigma_x(y)\) and \(\hat{\sigma}_x(y)\) there exist \(\hat{\phi} : X \to X\) such that for all \(a, b \in X\)
\[(a + b) \circ h = a \circ h + \hat{\phi}(h) + b \circ h.\]

D. The neutral element 0 of \((X, +)\) has a left and right distributivity.

Then for all \(a, b, c \in X\) the following statements hold:

1. \(\phi(a) = -a \circ 0\) and \(\hat{\phi}(h) = -0 \circ h\),
2. \(\sigma_a(b) = (a \circ b \circ z^{-1} - a \circ 0 + 1) \circ z = a \circ b - a \circ 0 \circ z + z.\)
3. \(a - a \circ 0 = 1\) and (i) \(0 \circ 0 = -1\) (ii) \(1 + 1 = 0^{-1}\).
4. If \(z_2 \circ \xi = 0^{-1}\), \(0 \circ \xi = z_1 \circ \xi = z^{-1} \circ 0^{-1}\), then (i)
\[\hat{\sigma}_a(b) \circ \hat{\tau}_b(a) = a \circ b = \sigma^z_a(b) \circ \tau^z_b(a)\]
(ii) \(-a \circ 0 + a = 1\).
**Remark.** Due to \( a - a \circ 0 = -a \circ 0 + a = 1 \), \( \forall a \in B \), we deduce that \( a + 1 = 1 + a \), \( \forall a \in B \).

**Definition**

A *near brace* is a set \( B \) together with two group operations \(+, \circ : B \times B \rightarrow B\), the first is called addition and the second is called multiplication, such that \( \forall a, b, c \in B \),

\[
a \circ (b + c) = a \circ b - a \circ 0 + a \circ c,
\]

and \( a - a \circ 0 = -a \circ 0 + a = 1 \). We denote by \( 0 \) the neutral element of the \((B, +)\) group and by \( 1 \) the neutral element of the \((B, \circ)\) group. We say that a near brace \( B \) is an abelian near brace if \( + \) is abelian.

**Example**

Let \((B, \circ)\) be a group with neutral element 1 and define \( a + b := a \circ \kappa^{-1} \circ b \), where \( 1 \neq \kappa \in B \) is an element of the center of \((B, \circ)\). Then \((B, \circ, +)\) is a near brace with neutral element \( 0 = \kappa \), and we call it the trivial near brace. Thanks to Paola!
Parametric maps & solutions of the braid equation

**Proposition**

Let \((B, \circ, +)\) be a near brace and let us denote \(\sigma^p_a(b) := a \circ b \circ z_1 - a \circ \xi + z_2\) and \(\tau^p_b(a) := \sigma^p_a(b)^{-1} \circ a \circ b\), where \(a, b \in B\), and \(h \in \{\xi, z_i\} \in B, i \in \{1, 2\}\) are fixed parameters, such that \(\exists c_1, c_2 \in B, \forall a, b, c \in B\), \((a - b + c) \circ h = a \circ h - b \circ h + c \circ h\), \(a \circ z_2 \circ z_1 - a \circ \xi = c_1\) and \(-a \circ \xi + a \circ z_1 \circ z_2 = c_2\). Then \(\forall a, b, c \in B\) the following properties hold:

1. \(\sigma^p_a(b) \circ \tau^p_b(a) = a \circ b\).
2. \(\sigma^p_a(\sigma^p_b(c)) = a \circ b \circ c \circ z_1 \circ z_1 - a \circ b \circ \xi \circ z_1 + c_1 + z_2\).
3. \(\sigma^p_a(b) \circ \sigma^p_{\tau^p_b(a)}(c) = a \circ b \circ c \circ z_1 + c_2 - a \circ \xi \circ z_2 + z_2 \circ z_2\).
Example
A simple example of the above generic maps is the case where $z_1 \circ z_2 = \xi \circ 0 = z_2 \circ z_1$, then $c_1 = c_2 = 1$.

Having showed the fundamental properties we prove the following theorem.

Theorem
Let $(B, \circ, +)$ be a near brace and $z \in B$ such that $\exists c_{1,2}, \forall a, b, c \in B$, $(a - b + c) \circ z_i = a \circ z_i - b \circ z_i + c \circ z_i$, $i \in \{1, 2\}$, $a \circ z_2 \circ z_1 - a \circ \xi = c_1$. We define a map $\tilde{r} : B \times B \rightarrow B \times B$ given by

$$\tilde{r}(a, b) = (\sigma^P_a(b), \tau^P_b(a)),$$

where $\sigma^P_a(b) = a \circ b \circ z_1 - a \circ \xi + z_2$, $\tau^P_b(a) = \sigma^P_a(b)^{-1} \circ a \circ b$. The pair $(B, \tilde{r})$ is a solution of the braid equation.

The inverse: $\tilde{r}^* : (x, y) \mapsto (\hat{\sigma}^P_x(y), \hat{\tau}^P_y(x))$, $\hat{\sigma}^P_x(y) = \hat{z}_2 - x \circ \hat{\xi} + x \circ y \circ \hat{z}_1$, $\hat{\tau}^P_x(y) = \hat{\sigma}^P_x(y)^{-1} \circ x \circ y$, where $\hat{\xi} = \xi^{-1}$, $\hat{z}_{1,2} = z_{1,2} \circ \xi^{-1}$. 

Anastasia Doikou Near braces, $p$-deformed braided groups & quasi-bialgebras
Motivated by Lu, Yan, & Zhu:

**Definition**

Let \((G, \circ)\) be a group, \(m(x, y) = x \circ y\) and \(\tilde{r}\) is an invertible map \(\tilde{r} : G \times G \to G \times G\), such that \(\forall x, y \in G, \tilde{r}(x, y) = (\sigma^p_x(y), \tau^p_y(x))\), where \(\sigma^p_x, \tau^p_y\) are bijective maps in \(G\). The map \(\tilde{r}\) is called a \(p\)-braiding operator (and the group is called \(p\)-braided) if

1. \(x \circ y = \sigma^p_x(y) \circ \tau^p_y(x)\).
2. \((\text{id} \times m) \tilde{r}_{12} \tilde{r}_{23}(x, y, w) = (f^p_{x \circ y}(w), f^p_{x \circ y}(w)^{-1} \circ x \circ y \circ w)\).
3. \((m \times \text{id}) \tilde{r}_{23} \tilde{r}_{12}(x, y, w) = (g^p_x(y \circ w), g^p_x(y \circ w)^{-1} \circ x \circ y \circ w)\).

for some bijections \(f^p_x, g^p_x : G \to G\), given \(\forall x \in G\).

**Proposition**

Let \((G, \circ)\) be a group, and the invertible map \(\tilde{r} : G \times G \to G \times G, \forall x, y \in G, \tilde{r}(x, y) = (\sigma^p_x(y), \tau^p_y(x))\), be a \(p\)-braiding operator for the group \(G\). Then \(\tilde{r}\) is a non-degenerate solution of the braid equation.

**Lemma**

Let \(B\) be a near brace, and consider the map \(\tilde{r} : B \times B \to B \times B, \tilde{r}(x, y) = (\sigma^p_x(y), \tau^p_y(x))\) of Proposition 1. Then \(\tilde{r}\) is a \(p\)-braiding.
Let $X$ be a set with $n$ elements and $\tilde{\rho}_z : X \times X \rightarrow X \times X$ be a set-theoretic solution of braid equation. Consider a free vector space $V = \mathbb{C}X$ of dimension equal to the cardinality of $X$. Let $\mathcal{B} = \{e_x\}_{x \in X}$ be the basis of $V$ and $\mathcal{B}^* = \{e^*_x\}_{x \in X}$ be the dual basis: $e^*_x e_y = \delta_{x,y}$.

Let $f \in V \otimes V$ be $f = \sum_{x,y \in X} f(x,y) e_x \otimes e_y$, then $r_z : V \otimes V \rightarrow V \otimes V$, such that $(\tilde{\rho}_z f)(x,y) = f(\sigma^z_x(y), \tau^z_y(x)), \forall x, y \in X$.

The set-theoretic solution

$$\tilde{\rho} = \sum_{x,y \in X} e_x,\sigma_x(y) \otimes e_y,\tau_y(x)$$

where $e_{x,y}$ are defined as $e_{x,y} = e_x e^*_y$ and $e^*_x = e^T$ ($^T$ denotes transposition). $e_{x,y}$ is an $n \times n$ matrix with one identity entry in $x$-row and $y$-column, and zeros elsewhere, i.e. $(e_{x,y})_{z,w} = \delta_{x,z} \delta_{y,w}$. 
The quantum algebra from skew braces

- **FRT (Faddeev, Reshetikhin, Takhtajan) construction.** Recall that given a solution of YBE, the quantum $A$ algebra is defined:

**Fundamental algebraic relation**

$$\tilde{r}_{12} L_1 L_2 = L_1 L_2 \tilde{r}_{12}$$

$$\tilde{r} \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N), \ L \in \text{End}(\mathbb{C}^N) \otimes A.$$  

- Recall *index notation*: $\tilde{r}_{12} = \tilde{r} \otimes 1_A$

$$L_1 = \sum_{z,w \in X} e_{z,w} \otimes I \otimes L_{z,w}, \quad L_2 = \sum_{z,w \in X} I \otimes e_{z,w} \otimes L_{z,w}$$

**The algebra**

From the FRT relation:

$$L_{x,\hat{x}}L_{y,\hat{y}} = L_{\sigma_x(y),\sigma_{\hat{x}}(\hat{y})}L_{\tau_y(x),\tau_{\hat{y}}(\hat{x})}.$$  

[Etingof, Shedler & Soloviev]

- The above algebra misleading! Alternative algebra: *(quasi)-triangular Hopf algebra*? No!...
Given a solution of YBE, another quadratic algebra \( Q \) algebra is defined:

### Quadratic algebra

\[
r_{12} q_1 q_2 = q_2 q_1
\]

\( r \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N), \ q \in \mathbb{C}^N \otimes Q \ (r = \mathcal{P} \mathcal{r}). \)

- Recall *index notation*: \( r_{12} = r \otimes 1_A \)

\[
q_1 = \sum_{z,w \in X} e_x \otimes l \otimes q_x, \quad q_2 = \sum_{x \in X} l \otimes e_x \otimes q_x
\]

### The algebra

From the quadratic algebra:

\[
q_x q_y = q_{\sigma_x(y)} q_{\tau_y(x)}.
\]

*[Etingof, Shedler & Soloviev]*

- The elements \( q_x \ \forall x \in X \) satisfy the \( Q \) algebraic relations and lead to a quasi-bialgebra!
Quasi-bialgebras

**Definition**

[Drinfeld]. A quasi-bialgebra \((\mathcal{A}, \Delta, \epsilon, \Phi, c_r, c_l)\) is a unital associative algebra \(\mathcal{A}\) over some field \(k\) with the following algebra homomorphisms:

- the co-product \(\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}\)
- the co-unit \(\epsilon : \mathcal{A} \rightarrow k\)

Together with the invertible element \(\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}\) (the associator) and the invertible elements \(c_l, c_r \in \mathcal{A}\) (unit constraints), such that:

1. \((\text{id} \otimes \Delta)\Delta(a) = \Phi\left( (\Delta \otimes \text{id})\Delta(a) \right) \Phi^{-1}, \forall a \in \mathcal{A}.

2. \(\left( (\text{id} \otimes \text{id} \otimes \Delta)\Phi \right) \left( (\Delta \otimes \text{id} \otimes \text{id})\Phi \right) = \left( (\text{id} \otimes \text{id} \otimes \text{id})\Phi \right) \left( (\text{id} \otimes \Delta \otimes \text{id})\Phi \right) \left( \Phi \otimes 1 \right).

3. \((\epsilon \otimes \text{id})\Delta(a) = c_l^{-1}ac_l\) and \((\text{id} \otimes \epsilon)\Delta(a) = c_r^{-1}ac_r, \forall a \in \mathcal{A}.

4. \((\text{id} \otimes \epsilon \otimes \text{id})\Phi = c_r \otimes c_l^{-1} \).
Quasi-triangular quasi-bialgebras

Recall, let \( A = \sum_j a_j \otimes b_j \in A \otimes A \), then in the “index” notation we denote:
\[
A_{12} := \sum_j a_j \otimes b_j \otimes 1, \quad A_{23} := \sum_j 1 \otimes a_j \otimes b_j \quad \text{and} \quad A_{13} := \sum_j a_j \otimes 1 \otimes b_j.
\]

Let \( \sigma \) be “flip” map, such that \( a \otimes b \mapsto b \otimes a \ \forall a, b \in A \), then \( \Delta^{(op)} := \sigma \circ \Delta \).

Definition

A quasi-bialgebra \((A, \Delta, \epsilon, \Phi)\) is called quasi-triangular (or braided) if an invertible element \( R \in A \otimes A \) (universal \( R \)-matrix) exists, such that

1. \( \Delta^{(op)}(a)R = R\Delta(a), \ \forall a \in A \).
2. \( (\text{id} \otimes \Delta)R = \Phi_{231}^{-1} R_{13} \Phi_{213} R_{12} \Phi_{123}^{-1} \).
3. \( (\Delta \otimes \text{id})R = \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi_{123} \).
We deduce: \((\epsilon \otimes \text{id})\mathcal{R} = c_r^{-1}c_l\) and \((\text{id} \otimes \epsilon)\mathcal{R} = c_l^{-1}c_r\), and \(\mathcal{R}\) satisfies a non-associative version of the Yang-Baxter equation

\[
\mathcal{R}_{12}\Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi_{123} = \Phi_{321}\mathcal{R}_{23}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}.
\]

For \(\Phi = 1 \otimes 1 \otimes 1\) \((c_r = c_l = 1)\) one recovers the quasi-triangular bialgebra and the YBE!

The main setup introduced in the following proposition will be central in proving key properties in the context of set-theoretic solutions.
Drinfeld twists

- Usually in literature the constraint \((\epsilon \otimes \text{id}) F = (\text{id} \otimes \epsilon) F = 1_A\) holds! We relax this constraint...Next a generalization of Drinfeld’s findings [AD, Ghionis & Vlaar]:

**Proposition**

Let \((A, \Delta, \epsilon, \Phi, R)\) be a quasi-triangular quasi-bialgebra and let \(F \in A \otimes A\) be an invertible element, such that

\[
\Delta_F(a) = F \Delta(a) F^{-1}, \quad \forall a \in A
\]

\[
\Phi_F(F \otimes 1)((\Delta \otimes \text{id}) F) = (1 \otimes F)((\text{id} \otimes \Delta) F) \Phi
\]

\[
R_F = F^{(op)} R F^{-1},
\]

where \(F^{(op)} := \sigma(F)\). Then \((A, \Delta_F, \epsilon, \Phi_F, R_F)\) is also a quasi-triangular quasi-bialgebra.

- Prove the axioms of the quasi-triangular quasi-bialgebrac for the generic scenario: \(F = \sum_j f_j \otimes g_j\). Let \(v := \sum_j \epsilon(f_j) g_j\), \(w := \sum_j \epsilon(g_j) f_j\):

\[
(\epsilon \otimes \text{id}) F = v, \quad (\text{id} \otimes \epsilon) F = w.
\]
Lemma (Main Frame)

Let \((A, \Delta, \epsilon, \Phi, R)\) and \((A, \Delta_F, \epsilon, \Phi_F, R_F)\) be quasi-triangular quasi-bialgebras and let the conditions of the Proposition hold. We recall also: \(F_{1,23} := (id \otimes \Delta)F\), \(F_{12,3} := (\Delta \otimes id)F\), then \(F_{21,3}R_{12} = R_{12}F_{12,3}\), \(F_{1,32}R_{23} = R_{23}F_{1,23}\). According to the Proposition we distinguish two cases:

1. If the associators satisfy

\[
\Phi_{213}R_{12} = R_{12}\Phi_{123}, \quad \Phi_F213R_{12} = R_{12}\Phi_F123, \quad (4)
\]

and \(\Phi_F\) commutes with \(F_{12}\), then the condition (4) can be re-expressed as

\[
F_{123} := F_{23}F_{1,23} = F_{12}F_{12,3}^*, \text{ where } F_{12,3}^* = \Phi_FF_{12,3}\Phi^{-1}. \text{ We also deduce that }
\]

\[
F_{21,3}^*R_{12} = R_{12}F_{21,3}^*,
\]

2. If the associator satisfies

\[
\Phi_{132}R_{23} = R_{23}\Phi_{123}, \quad \Phi_F132R_{23} = R_{23}\Phi_F123, \quad (5)
\]

and \(\Phi_F\) commutes with \(F_{23}\), then then condition (4) is re-expressed as

\[
F_{123} := F_{23}F_{1,23}^* = F_{12}F_{12,3}, \text{ where } F_{12,3}^* = \Phi^{-1}_F123\Phi. \text{ We also deduce that }
\]

\[
F_{1,32}^*R_{23} = R_{23}F_{1,23}^*.
\]

[AD, Ghionis & Vlaar] & [AD, & Rybolowicz]
Twists & quasi-bialgebras from braces

Drinfeld Twist

[AD & Smoktunowicz], [AD]. Let \( \tilde{r} = \sum_{x,y\in\mathcal{X}} e_{x,\sigma(x)(y)} \otimes e_{y,\tau(y)(x)} \) be the set-theoretic solution of the braid YBE, \( P \) is the permutation operator and \( \hat{V}_k, V_k \) are their respective eigenvectors. Let \( F^{-1} = \sum_{k=1}^{n^2} \hat{V}_k V_k^T \) be the similarity transformation (twist), such that \( \tilde{r} = F^{-1}PF \). Then the twist can be explicitly expressed as \( F = \sum_{x\in\mathcal{X}} e_{x,x} \otimes V_x \), where we define \( V_x = \sum_{y\in\mathcal{X}} e_{\sigma(x)(y),y} \).

- The twist is not uniquely defined, for instance an alternative twist is \( \hat{F} = \sum_{x,y\in\mathcal{X}} e_{\tau(y)(x),x} \otimes e_{y,y} = \sum_y W_y \otimes e_{y,y} \).

- Note: The matrices \( V_x \) and \( W_x^T \) (\( T \) denotes transposition), are \( n \)-dimensional representations of \( Q \), i.e. \( q_x \mapsto U_x \), where \( U_x \in \{ V_x, W_x^T \} \).

- Both twists are still admissible in the case of non-involutive, invertible solutions.
The twisted $r$-matrix

Lemma

Let $\tilde{r} : V \otimes V \rightarrow V \otimes V$ be the set-theoretic solution of the braid equation. Let also $V_x = \sum_{y \in X} e_{\sigma(x)(y),y}$ and $W_x = \sum_{\eta \in X} e_{\tau(x)(\eta),\eta}$, $\forall x \in X$, with coproducts defined.

Then $\Delta_T(\mathbb{Y}_x)\tilde{r}_T = \tilde{r}_T \Delta_T(\mathbb{Y}_x)$, where $\mathbb{Y}_x \in \{V_x, W_x\}$, $T \in \{F, \hat{F}\}$,

$$
\Delta_F(V_x) = V_x \otimes V_x, \quad \Delta_F(W_y) = \sum_{\eta, x \in X} e_{\tau(x)(\eta),\eta} \otimes e_{\tau(x)(\eta),\eta}(\sigma(\eta)(x),\sigma(\eta)(x))
$$

$$
\Delta_{\hat{F}}(V_x) = \sum_{x, y \in X} e_{\sigma(x)(y),\sigma(x)(y)}(\tau(x)(y),\tau(x)(y)) \otimes e_{\sigma(x)(y),\sigma(x)(y)}(y) \otimes e_{\sigma(x)(y),\sigma(x)(y)}(y) \otimes e_{\sigma(x)(y),\sigma(x)(y)}(y)
$$

and the twisted matrices read as

$$
\tilde{r}_F = \sum_{x, y \in X} e_{\sigma(x)(y),\sigma(x)(y)} \otimes e_{\sigma(x)(y),\sigma(x)(y)}(\tau(x)(y)) \quad \& \quad \tilde{r}_{\hat{F}} = \sum_{x, y \in X} e_{\tau(x)(y),\tau(x)(y)}(\sigma(x)(y)) \otimes e_{\tau(x)(y),\tau(x)(y)}(y).
$$

[AD & Rybolowicz]

Involutive case: $\tilde{r}_F = \tilde{r}_{\hat{F}} = \mathcal{P}$. Baxterization $\tilde{R}(\lambda) = \lambda \tilde{r} + \mathcal{P}$ and relation with Yangian [AD, Ghionis & Vlaar] (extended the results of [Etingof, Shedler & Soloviev]).
Focus on $\check{r}_F = \sum_{x,y \in X} e_x \otimes e_y \cdot y \triangleright x$, where $y \triangleright x = \sigma_y(\check{\sigma}_y^{-1}(x))$ and $(X, \triangleright)$ is a rack, i.e. $\triangleright$ is self distributive. (Relations to racks and quandles)

Quadratic algebra from $r_F(q \otimes 1)(1 \otimes q) = (1 \otimes q)(q \otimes 1)$:
$q_x q_y = q_y q_y \triangleright x$. 

Anastasia Doikou
Near braces, $p$-deformed braided groups & quasi-bialgebras
- Study of the quantum group associated to the derived solution....Hopf-algebra?
- Solve eigenvalue problem for non-involutive solutions of the YBE. Challenging problem that will provide valuable info on multiplicities and hence associated symmetries.

**Ultimate goal:** Identify spectrum and eigenstates of local Hamiltonians (quantum systems) constructed from set-theoretic solutions. Key point: *Use of Drinfeld twists!* Use info gained from the study of symmetries: *spectrum multiplicities.*

- Next challenging problem: universal $R$-matrix!