Near braces, p-deformed braided groups & quasi-bialgebras

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- *A.D., J. Phys. A54 (2021) 415201*
  A.D., A. Ghionis & B. Vlaar, Lett. Math. Phys. (2022)
  A.D. & B. Rybolowicz, arXiv:2204.11580 (2022)
  A.D. & B. Rybolowicz, arXiv:2204.11580 (2023)

Review YBE

- YBE introduced: *Yang*, study of N particle in δ potential & *Baxter*, study of XYZ model.
- YBE and the R-matrix are the quantum analogues of the classical YBE and the classical r-matrix (Sklyanin bracket) [*Semenof-Tjian-Shanski Sklyanin ....*], classical intergrable systems.
- Fundamental equ. in QISM formulation [*Faddeev, Tahktajan, Kulish, Sklyanin, Reshetikhin....*] & quantum algebras (deformed Lie algebras) [*Drinfeld, Jimbo*].
- Solving YBE: e.g. using braids and Hecke algebras *Baxterization* [*Jimbo*], or using linear intertwining relations (quantum group symmetry) [*Kulish*].

Review

- [*Drinfeld*] introduced the "Set-theoretic YBE".
- [*Hietarinta*] first classification of set-theoretic solutions of YBE. [*Etingof, Shedler & Soloviev*] set theoretical solutions & quantum groups for param. free R-matrices.
- Connections to: geometric crystals [*Berenstein & Kazhdan, Etingof*] and cellular automatons [*Hatayama, Kuniba & Takagi*].
- Classical discrete integrable systems (YB maps), quad-graph, discrete maps: [*Veselov, Bobenko, Suris, Papageorgiou, Tongas,...*]
- Set theoretical involutive solutions of YBE from **braces**: [*Rump, Guarnieri, Vendramin, Gateva-Ivanova, Cedó, Jespers, Okniński, Smoktunowicz,...*]

Outline

1. We recall basic definitions and results about braces and set-theoretic solutions of the Yang-Baxter equation. We introduce slightly generalized algebraic structures called *near-braces* associated to solutions of the YBE. Starting from set theoretic solutions with extra fixed parameters, we reconstruct the near brace (*AD, Rybolowicz*) and vise versa. The notion of the p-deformed braided groups is also introduced leading to generalized solutions of the YBE.
2. We recall the definitions of the quasi bi-algebra and we then move on to study the quantum algebra associated to set-theoretic solutions of the YBE (*Etingof, Shedler, Soloviev*). The quantum algebra is a quasi bialgebra & the set-theoretic solutions are obtained via an admissible Drinfeld twist from a derived solution (*AD, Ghionis, Vlaar; AD, Rybolowicz; AD*).

Set-theoretic solutions of thel YBE

- Let a set $X = \{x_1, \dots, x_{\mathcal{N}}\}$ and $\check{r} : X \times X \to X \times X$. Denote

$$\check{r}(x, y) = (\sigma_x(y), \tau_y(x))$$

  1. $(X, \check{r})$ non-degenerate: σ_x and τ_y are bijective functions
  2. $(X, \check{r})$ involutive: $\check{r}(\sigma_x(y), \tau_y(x)) = (x, y)$, $(\check{r}\check{r}(x, y) = (x, y))$

- *Suppose* $(X, \check{r})$ *is an involutive, non-degenerate set-theoretic solution of the Braid equation*:

Braid equation

$$(\check{r} \times Id_X)(Id_X \times \check{r})(\check{r} \times Id_X) = (Id_X \times \check{r})(\check{r} \times Id_X)(Id_X \times \check{r}).$$

YBE solutions from braces

Definition (Rump; Guarnieri & Vendramin)

A *left skew brace* is a set B together with two group operations $+, \circ : B \times B \to B$, the first is called addition and the second is called multiplication, such that $\forall a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c.$$

If $+$ is an abelian group operation B is called a *left brace*. Moreover, if B is a left skew brace and $\forall a, b, c \in B$ $(b + c) \circ a = b \circ a - a + c \circ a$, then B is called a *skew brace*. Analogously if $+$ is abelian and B is a skew brace, then B is called a *brace*.

- The additive identity of a left skew brace B will be denoted by 0 and the multiplicative identity by 1. In every left skew brace $0 = 1$.

YBE solutions from braces

Theorem (*Rump*)

Let $(B, +, \circ)$ be a brace and for $x, y \in B$ define $\check{r}(x, y) = (\sigma_x(y),\ \tau_y(x))$, the YB map of $(B, +, \circ)$:

$$\sigma_x(y) = x \circ y - x, \quad \tau_y(x) = (\sigma_x(y))^{-1} \circ x - (\sigma_x(y))^{-1}.$$

Then (B, r) is a non-degenerate, involutive solution of the set-theoretic YBE. Conversely, for every non-degenerate, involutive solution of the set-theoretic YBE $(X, \check{r})$ there is a brace $(B, +, \circ)$: $X \subseteq B$ and $\check{r}$ is the restriction of the YB map of $(B, +, \circ)$ to $X \times X$.

- See also [Cedo, Jespers, Okninski].
- Guarnieri & Vendramin generalized Rump's results for non-involutive, non-degenerate, set-theoretic solutions using skew braces.

Extended non-involutive solutions from (skew) braces

- A generalized version of the set-theoretic solution via some kind of "z-deformation". Let $z \in X$ be fixed, then we denote

$$\check{r}_z(x,y) = (\sigma^z_x(y), \tau^z_y(x)).$$

  $\check{r}$ is non-degenerate if σ^z_x and τ^z_y are bijective maps.

- By requiring $(X, \check{r}_z)$ to be a solution of the braid equation:

$$(\check{r} \times \mathrm{id})(\mathrm{id} \times \check{r})(\check{r} \times \mathrm{id}) = (\mathrm{id} \times \check{r})(\check{r} \times \mathrm{id})(\mathrm{id} \times \check{r}),$$

  we obtain the fundamental constraints:

$$\sigma^z_\eta(\sigma^z_x(y)) = \sigma^z_{\sigma^z_\eta(x)}(\sigma^z_{\tau^z_x(\eta)}(y)), \qquad (1)$$

$$\tau^z_y(\tau^z_x(\eta)) = \tau^z_{\tau^z_y(x)}(\tau^z_{\sigma^z_x(y)}(\eta)), \qquad (2)$$

$$\tau^z_{\sigma^z_{\tau^z_x(\eta)}(y)}(\sigma^z_\eta(x)) = \sigma^z_{\tau^z_{\sigma^z_x(y)}(\eta)}(\tau^z_y(x)). \qquad (3)$$

Extended non-involutive solutions from (skew) braces

Theorem

Let B be a left skew brace and $z \in B$ such that $\forall a, b, c \in B$, $(a - b + c) \circ z = a \circ z - b \circ z + c \circ z$. Then we can define a map $\check{r}_z : B \times B \to B \times B$ given by

$$\check{r}_z(a, b) = (\sigma^z_a(b), \tau^z_b(a)) := (a \circ b - a \circ z + z,\ (a \circ b - a \circ z + z)^{-1} \circ a \circ b),$$

where $(a \circ b - a \circ z + z)^{-1}$ is the inverse in the group $(B, \circ)$. The pair $(B, \check{r})$ is a solution of the braid equation.

Proof. IShow that the maps σ^z, τ^z satisfy the constraints (1)-(3) [*AD & Rybolowicz*].

1. **Remark.** *We note that σ^z_x, τ^z_y as defined in the Theorem are bijections and thus $\check{r}$ is non-degenerate.*
2. **Remark.** *For $z = 1$, Guarnieri-Vedramin skew braces are recovered; if in addition $(B, +)$ is an abelian group Rump's braces are recovered and $\check{r}_{z=1}$ becomes involutive($\sigma^z_{\sigma^z_x(y)}(\tau^z_y(x)) = x$ and $\tau^z_{\tau^z_y(x)}(\sigma^z_x(y)) = y$).* **For $z \neq 1$ the solutions are not involutive in general, even in the case of braces.**
   ****σ_x is a group action but τ_y is not!!**

The inverse $\check{r}$ matrix

- The explicit expressions of the inverse $\check{r}_z$-matrices & the bijective maps:

Proposition

Let $\check{r}_z$, $\check{r}^*_{\hat{z}} : X \times X$ be solutions of the braid equations, such that
$\check{r}^* : (x, y) \mapsto (\hat{\sigma}^z_x(y), \hat{\tau}^z_y(x))$, $\check{r} : (x, y) \mapsto (\sigma^z_x(y), \tau^z_y(x))$.

1. $\check{r}^* = \check{r}^{-1}$ if and only if

$$\hat{\sigma}^z_{\sigma^z_x(y)}(\tau^z_y(x)) = x, \ \hat{\tau}^z_{\tau^z_y(x)}(\sigma^z_x(y)) = y, \ \sigma^z_{\hat{\sigma}^z_x(y)}(\hat{\tau}^z_y(x)) = x, \ \tau^z_{\hat{\tau}^z_y(x)}(\hat{\sigma}^z_x(y)) = y.$$

2. If $\sigma^z_x(y) = x \circ y - x \circ z + z$, $\tau^z_y(x) = \sigma^z_x(y)^{-1} \circ x \circ y$, then
$\hat{\sigma}^z_x(y) = -x \circ z^{-1} + x \circ y \circ z^{-1}$, $\hat{\tau}^z_x(y) = \hat{\sigma}^z_x(y)^{-1} \circ x \circ y$.

Examples

Example (1)

Let us consider a set $\mathrm{Odd} := \left\{ \frac{2n+1}{2k+1} \mid n, k \in \mathbb{Z} \right\}$ together with two binary operations $(a, b) \xrightarrow{+_1} a - 1 + b$ and $(a, b) \xrightarrow{\circ} a \cdot b$, where $+, \cdot$ are addition and multiplication of rational numbers, respectively. The triple $(\mathrm{Odd}, +_1, \circ)$ is a brace. The solution $\check{r}_z$ is involutive if and only if $a \cdot b - a \cdot z + z = a \cdot b - a + 1$ if and only if $(z - 1) \cdot (1 - a) = 0$, $\forall a, b \in B$. Therefore, $\forall z \neq 1$, $\check{r}_z$ is non-involutive. Moreover, $\check{r}_z = \check{r}_w$ if and only if $-a \cdot z + z = -a \cdot w + w$, that is if $z = w$.

Example (2)

Let us consider a ring $\mathbb{Z}/8\mathbb{Z}$. A triple

$$\left(\mathrm{OM} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, d \in \{1, 3, 5, 7\},\ b, c \in \{0, 2, 4, 6\} \right\}, +_{\mathbb{I}}, \circ \right)$$

is a brace, where $(A, B) \xrightarrow{+_{\mathbb{I}}} A - \mathbb{I} + B$, $(A, B) \xrightarrow{\circ} A \cdot B$, and $+, \cdot$ are addition and multiplication of two by two matrices over $\mathbb{Z}/8\mathbb{Z}$, respectively. Moreover one can easily check that two solutions $\check{r}_A$ and $\check{r}_B$ are equal if and only if $(D - \mathbb{I}) \cdot (B - A) = 0 \pmod 8$ $\forall D \in \mathrm{OM}$.

The examples above are inspired by works on trusses, paragons...: [Brzezinski], [Brzezinski & Rybolowicz].

Near-braces

- Let X be a set with a group operation $\circ : X \times X \to X$, with a neutral element $1 \in X$ and an inverse $x^{-1} \in X$, $\forall x \in X$. Let $\sigma^z_x : X \to X$, be a family of bijective functions: $y \mapsto \sigma^z_x(y)$, where $z \in X$ is some fixed parameter. We define another binary operation $+ : X \times X \to X$, such that

$$y + x := x \circ \sigma^z_{x^{-1}}(y \circ z) \circ z^{-1}$$

For convenience we will omit henceforth the fixed $z \in X$ in $\sigma^z_x(y)$. We assume that $+$ is associative.

Near-braces

- Focus on non-degenerate, invertible solutions $\check{r}$ and σ_x and τ_y are bijections:
$$\sigma_x^{-1}(\sigma_x(y)) = \sigma_x(\sigma_x^{-1}(y)) = y, \quad \tau_y^{-1}(\tau_y(x)) = \tau_y(\tau_y^{-1}(x)) = x$$
- Let $\check{r}^{-1}(x,y) = (\hat{\sigma}_x(y), \hat{\tau}_y(x))$ exist with $\hat{\sigma}_x$, $\hat{\tau}_y$ being also bijections:
$$\sigma_{\hat{\sigma}_x(y)}(\hat{\tau}_y(x)) = x = \hat{\sigma}_{\sigma_x(y)}(\tau_y(x)), \quad \tau_{\hat{\tau}_y(x)}(\hat{\sigma}_x(y)) = y = \hat{\tau}_{\tau_y(x)}(\sigma_x(y)).$$
$$\hat{\sigma}^{-1}_{\sigma_x(y)}(x) = \tau_y(x), \quad \hat{\tau}^{-1}_{\tau_y(x)}(y) = \sigma_x(y).$$
- Assume that $\hat{\sigma}$ has the general form
$$\hat{\sigma}_x(y) := x \circ (x^{-1} \circ z_2 + y \circ z_1) \circ \xi,$$
the parameters $z_{1,2}$, ξ to be identified.

Near-braces

- Useful Lemmas:

Lemma 1

For all $x \in X$, the operations $+x,\ x+ : X \to X$ are bijections.

- We now introduce the notion of a neutral elements in $(X, +)$

Lemma 2

Let $(X, +)$ be a semigroup, then $\forall x \in X$, $\exists\ 0_x \in X$, such that $0_x + x = x$. Moreover, $\forall x, y \in X$, $0_x = 0_y = 0$, i.e. 0 is the unique left neutral element. The left neutral element 0 is also right neutral element.

Lemma 3

Let 0 be the neutral element in $(X, +)$, then $\forall x \in X$, $\exists -x \in X$, such that $-x + x = 0$ (left inverse). Moreover, $-x \in X$ is a right inverse, i.e. $x + (-x) = 0$ $\forall x \in X$. That is $(X, +, 0)$ is a group.

Near-braces

Theorem

Let $(X, \circ)$ be a group and $\check{r} : X \times X \to X \times X$ be such that $\check{r}(x, y) = (\sigma_x(y), \tau_x(y))$ is a non-degenerate solution of the set-theoretic braid equation. Moreover, we assume that:

- (A) *The pair $(X, +)$ ($+$ as defined) is a group.*
- (B) *There exists $\phi : X \to X$ such that for all $a, b, c \in X$* $a \circ (b + c) = a \circ b + \phi(a) + a \circ c$.
- (C) *For $h \in \{z, \xi\} \in X$ appearing in $\sigma_x(y)$ and $\hat{\sigma}_x(y)$ there exist $\widehat{\phi} : X \to X$ such that for all $a, b \in X$* $(a + b) \circ h = a \circ h + \widehat{\phi}(h) + b \circ h$.
- (D) *The neutral element 0 of $(X, +)$ has a left and right distributivity.*

Then for all $a, b, c \in X$ the following statements hold:

1. $\phi(a) = -a \circ 0$ *and* $\widehat{\phi}(h) = -0 \circ h$,
2. $\sigma_a(b) = (a \circ b \circ z^{-1} - a \circ 0 + 1) \circ z = a \circ b - a \circ 0 \circ z + z$.
3. $a - a \circ 0 = 1$ *and (i)* $0 \circ 0 = -1$ *(ii)* $1 + 1 = 0^{-1}$.
4. *If* $z_2 \circ \xi = 0^{-1}$, $0 \circ \xi = z_1 \circ \xi = z^{-1} \circ 0^{-1}$, *then (i)* $\hat{\sigma}_a(b) \circ \hat{\tau}_b(a) = a \circ b = \sigma^z_a(b) \circ \tau^z_b(a)$ *(ii)* $-a \circ 0 + a = 1$.

Near-braces

- **Remark.** Due to $a - a \circ 0 = -a \circ 0 + a = 1$, $\forall a \in B$, we deduce that $a + 1 = 1 + a$, $\forall a \in B$.

Definition

A *near brace* is a set B together with two group operations $+, \circ : B \times B \to B$, the first is called addition and the second is called multiplication, such that $\forall a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a \circ 0 + a \circ c,$$

and $a - a \circ 0 = -a \circ 0 + a = 1$. We denote by 0 the neutral element of the $(B, +)$ group and by 1 the neutral element of the $(B, \circ)$ group. We say that a near brace B is an abelian near brace if $+$ is abelian.

- In the special case where $0 = 1$, we recover a skew brace.

Example

Let $(B, \circ)$ be a group with neutral element 1 and define $a + b := a \circ \kappa^{-1} \circ b$, where $1 \neq \kappa \in B$ is an element of the center of $(B, \circ)$. Then $(B, \circ, +)$ is a near brace with neutral element $0 = \kappa$, and we call it the trivial near brace. **Thanks to Paola!**

Parametric maps & solutions of the braid equation

Proposition

Let $(B, \circ, +)$ be a near brace and let us denote $\sigma_a^p(b) := a \circ b \circ z_1 - a \circ \xi + z_2$ and $\tau_b^p(a) := \sigma_a^p(b)^{-1} \circ a \circ b$, where $a, b \in B$, and $h \in \{\xi, z_i\} \in B$, $i \in \{1, 2\}$ are fixed parameters, such that $\exists\ c_{1,2} \in B, \forall a, b, c \in B$, $(a - b + c) \circ h = a \circ h - b \circ h + c \circ h$, $a \circ z_2 \circ z_1 - a \circ \xi = c_1$ and $-a \circ \xi + a \circ z_1 \circ z_2 = c_2$. Then $\forall a, b, c \in B$ the following properties hold:

1. $\sigma_a^p(b) \circ \tau_b^p(a) = a \circ b$.
2. $\sigma_a^p(\sigma_b^p(c)) = a \circ b \circ c \circ z_1 \circ z_1 - a \circ b \circ \xi \circ z_1 + c_1 + z_2$.
3. $\sigma_a^p(b) \circ \sigma^p_{\tau_b^p(a)}(c) = a \circ b \circ c \circ z_1 + c_2 - a \circ \xi \circ z_2 + z_2 \circ z_2$.

Parametric maps & solutions of the braid equation

Example

A simple example of the above generic maps is the case where $z_1 \circ z_2 = \xi \circ 0 = z_2 \circ z_1$, then $c_1 = c_2 = 1$.

Having showed the fundamental properties we prove the following theorem.

Theorem

Let $(B, \circ, +)$ be a near brace and $z \in B$ such that $\exists\ c_{1,2}, \forall a, b, c \in B$, $(a - b + c) \circ z_i = a \circ z_i - b \circ z_i + c \circ z_i$, $i \in \{1, 2\}$, $a \circ z_2 \circ z_1 - a \circ \xi = c_1$. We define a map $\check{r} : B \times B \to B \times B$ given by

$$\check{r}(a, b) = (\sigma^p_a(b), \tau^p_b(a)),$$

where $\sigma^p_a(b) = a \circ b \circ z_1 - a \circ \xi + z_2$, $\tau^p_b(a) = \sigma^p_a(b)^{-1} \circ a \circ b$. The pair $(B, \check{r})$ is a solution of the braid equation.

The inverse: $\check{r}^* : (x, y) \mapsto (\hat{\sigma}^p_x(y), \hat{\tau}^p_y(x))$, $\hat{\sigma}^p_x(y) = \hat{z}_2 - x \circ \hat{\xi} + x \circ y \circ \hat{z}_1$, $\hat{\tau}^p_x(y) = \hat{\sigma}^p_x(y)^{-1} \circ x \circ y$, where $\hat{\xi} = \xi^{-1}$, $\hat{z}_{1,2} = z_{1,2} \circ \xi^{-1}$.

p-deformed braided groups and near braces

Motivated by Lu, Yan, & Zhu:

Definition

Let $(G, \circ)$ be a group, $m(x,y) = x \circ y$ and $\check{r}$ is an invertible map $\check{r} : G \times G \to G \times G$, s uch that $\forall x, y \in G$, $\check{r}(x,y) = (\sigma^p_x(y), \tau^p_y(x))$, where σ^p_x, τ^p_y are bijective maps in G. The map $\check{r}$ is called a p-braiding operator (and the group is called p-braided) if

1. $x \circ y = \sigma^p_x(y) \circ \tau^p_y(x)$.
2. $(\mathrm{id} \times m)\ \check{r}_{12}\ \check{r}_{23}(x,y,w) = (f^p_{x\circ y}(w),\ f^p_{x \circ y}(w)^{-1} \circ x \circ y \circ w)$.
3. $(m \times \mathrm{id})\ \check{r}_{23}\ \check{r}_{12}(x,y,w) = (g^p_x(y \circ w),\ g^p_x(y \circ w)^{-1} \circ x \circ y \circ w)$.

for some bijections $f^p_x, g^p_x : G \to G$, given $\forall x \in G$.

Proposition

Let $(G, \circ)$ be a group, and the invertible map $\check{r} : G \times G \to G \times G$, $\forall x, y \in G$, $\check{r}(x,y) = (\sigma^p_x(y), \tau^p_y(x))$, be a p-braiding operator for the group G. Then $\check{r}$ is a non-degenerate solution of the braid equation.

Lemma

Let B be a near brace, and consider the map $\check{r} : B \times B \to B \times B$, $\check{r}(x,y) = (\sigma^p_x(y),\ \tau^p_y(x))$ of Proposition 1. Then $\check{r}$ is a p-braiding.

Set-theoretic solutions, matrix notation

- Let X be a set with n elements and $\check{r}_z : X \times X \to X \times X$ be a set-theoretic solution of braid equation. Consider a free vector space $V = \mathbb{C}X$ of dimension equal to the cardinality of X. Let $\mathbb{B} = \{e_x\}_{x \in X}$ be the basis of V and $\mathbb{B}^* = \{e_x^*\}_{x \in X}$ be the dual basis: $e_x^* e_y = \delta_{x,y}$.
- Let $f \in V \otimes V$ be $f = \sum_{x,y \in X} f(x,y) e_x \otimes e_y$, then $r_z : V \otimes V \to V \otimes V$, such that $(\check{r}_z f)(x,y) = f(\sigma_x^z(y), \tau_y^z(x))$, $\forall x, y \in X$:

The set-theoretic solution

$$\check{r} = \sum_{x,y \in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$$

where $e_{x,y}$ are defined as $e_{x,y} = e_x e_y^*$ and $e_x^* = e^T$ (^{T} denotes transposition). $e_{x,y}$ is an $n \times n$ matrix with one identity entry in x-row and y-column, and zeros elsewhere, i.e $(e_{x,y})_{z,w} = \delta_{x,z}\delta_{y,w}$.

The quantum algebra from skew braces

- *FRT (Faddeev, Reshetikhin, Takhtajan) construction*. Recall that given a solution of YBE, the quantum $\mathcal{A}$ algebra is defined:

Fundamental algebraic relation

$$\check{r}_{12}\ L_1\ L_2 = L_1\ L_2\ \check{r}_{12}$$

$\check{r} \in \text{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}}), \ \ L \in \text{End}(\mathbb{C}^{\mathcal{N}}) \otimes \mathcal{A}.$

- Recall *index notation*: $\check{r}_{12} = \check{r} \otimes 1_{\mathcal{A}}$

$$L_1 = \sum_{z,w\in X} e_{z,w} \otimes I \otimes L_{z,w}, \qquad L_2 = \sum_{z,w\in X} I \otimes e_{z,w} \otimes L_{z,w}$$

The algebra

From the FRT relation:

$$L_{x,\hat{x}} L_{y,\hat{y}} = L_{\sigma_x(y),\sigma_{\hat{x}}(\hat{y})} L_{\tau_y(x),\tau_{\hat{y}}(\hat{x})}.$$

[*Etingof, Shedler & Soloviev*]

- The above algebra **misleading!** Alternative algebra: **(quasi)-triangular Hopf algebra**? No!...

Another quadratic algebra from skew braces

- Given a solution of YBE, another quadratic algebra $\mathcal{Q}$ algebra is defined:

Quadratic algebra

$$r_{12}\ q_1\ q_2 = q_2\ q_1$$

$r \in \text{End}(\mathbb{C}^{\mathcal{N}} \otimes \mathbb{C}^{\mathcal{N}}),\ \ q \in \mathbb{C}^{\mathcal{N}} \otimes \mathcal{Q}\ (r = \mathcal{P}\check{r}).$

- Recall *index notation*: $r_{12} = r \otimes 1_{\mathcal{A}}$

$$q_1 = \sum_{z,w \in X} e_x \otimes I \otimes q_x, \qquad q_2 = \sum_{x \in X} I \otimes e_x \otimes q_x$$

The algebra

From the quadratic algebra:

$$q_x q_y = q_{\sigma_x(y)} q_{\tau_y(x)}.$$

[*Etingof, Shedler & Soloviev*]

- The elements q_x $\forall x \in X$ satisfy the $\mathcal{Q}$ algebraic relations and lead to a quasi-bialgebra!

Quasi-bialgebras

Definition

[*Drineld*]. A quasi-bialgebra $\left(\mathcal{A}, \Delta, \epsilon, \Phi, c_r, c_l\right)$ is a unital associative algebra $\mathcal{A}$ over some field k with the following algebra homomorphisms:

- the co-product $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$
- the co-unit $\epsilon : \mathcal{A} \to k$

together with the invertible element $\Phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ (the associator) and the invertible elements $c_l, c_r \in \mathcal{A}$ (unit constraints), such that:

1. $(\mathrm{id} \otimes \Delta)\Delta(a) = \Phi\Big((\Delta \otimes \mathrm{id})\Delta(a)\Big)\Phi^{-1}, \ \forall a \in \mathcal{A}.$
2. $\Big((\mathrm{id} \otimes \mathrm{id} \otimes \Delta)\Phi\Big)\Big((\Delta \otimes \mathrm{id} \otimes \mathrm{id})\Phi\Big) = \Big(1 \otimes \Phi\Big)\Big((\mathrm{id} \otimes \Delta \otimes \mathrm{id})\Phi\Big)\Big(\Phi \otimes 1\Big).$
3. $(\epsilon \otimes \mathrm{id})\Delta(a) = c_l^{-1} a c_l$ and $(\mathrm{id} \otimes \epsilon)\Delta(a) = c_r^{-1} a c_r, \ \forall a \in \mathcal{A}.$
4. $(\mathrm{id} \otimes \epsilon \otimes \mathrm{id})\Phi = c_r \otimes c_l^{-1}.$

Quasi-triangular quasi-bialgebras

- Recall, let $A = \sum_j a_j \otimes b_j \in \mathcal{A} \otimes \mathcal{A}$, then in the "index" notation we denote: $A_{12} := \sum_j a_j \otimes b_j \otimes 1$, $A_{23} := \sum_j 1 \otimes a_j \otimes b_j$ and $A_{13} := \sum_j a_j \otimes 1 \otimes b_j$.
- Let σ be "flip" map, such that $a \otimes b \mapsto b \otimes a \; \forall a, b \in \mathcal{A}$, then $\Delta^{(op)} := \sigma \circ \Delta$.

Definition

A quasi-bialgebra $\big(\mathcal{A}, \Delta, \epsilon, \Phi\big)$ is called quasi-triangular (or braided) if an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ (universal R-matrix) exists, such that

1. $\Delta^{(op)}(a)\mathcal{R} = \mathcal{R}\Delta(a), \; \forall a \in \mathcal{A}$.
2. $(\mathrm{id} \otimes \Delta)\mathcal{R} = \Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}\Phi_{123}^{-1}$.
3. $(\Delta \otimes \mathrm{id})\mathcal{R} = \Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi_{123}$.

Quasi-bialgebras

- We deduce: $(\epsilon \otimes \mathrm{id})\mathcal{R} = c_r^{-1} c_l$ and $(\mathrm{id} \otimes \epsilon)\mathcal{R} = c_l^{-1} c_r$, and $\mathcal{R}$ satisfies a non-associative version of the Yang-Baxter equation

$$\mathcal{R}_{12}\Phi_{312}\mathcal{R}_{13}\Phi_{132}^{-1}\mathcal{R}_{23}\Phi_{123} = \Phi_{321}\mathcal{R}_{23}\Phi_{231}^{-1}\mathcal{R}_{13}\Phi_{213}\mathcal{R}_{12}.$$

- For $\Phi = 1 \otimes 1 \otimes 1$ ($c_r = c_l = 1$) one recovers the quasi-triangular bialgebra and the YBE!
- The main setup introduced in the following proposition will be central in proving key properties in the context of set-theoretic solutions.

Drinfeld twists

- Usually in literature the constraint $(\epsilon \otimes \text{id})\mathcal{F} = (\text{id} \otimes \epsilon)\mathcal{F} = 1_{\mathcal{A}}$ holds! We relax this constraint...Next a generalization of Drinfeld's findings *[AD, Ghionis & Vlaar]*:

Proposition

Let $(\mathcal{A}, \Delta, \epsilon, \Phi, \mathcal{R})$ be a quasi-triangular quasi-bialgebra and let $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$ be an invertible element, such that

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}, \quad \forall a \in \mathcal{A}$$
$$\Phi_{\mathcal{F}}(\mathcal{F} \otimes 1)\big((\Delta \otimes \text{id})\mathcal{F}\big) = (1 \otimes \mathcal{F})\big((\text{id} \otimes \Delta)\mathcal{F}\big)\Phi$$
$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}^{(op)}\mathcal{R}\mathcal{F}^{-1},$$

where $\mathcal{F}^{(op)} := \sigma(\mathcal{F})$. Then $(\mathcal{A}, \Delta_{\mathcal{F}}, \epsilon, \Phi_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ is also a quasi-triangular quasi-bialgebra.

- Prove the axioms of the quasi-triangular quasi-bialgebrac for the generic scenario: $\mathcal{F} = \sum_j f_j \otimes g_j$. Let $v := \sum_j \epsilon(f_j)g_j, \;\; w := \sum_j \epsilon(g_j)f_j$:

$$(\epsilon \otimes \text{id})\mathcal{F} = v, \quad (\text{id} \otimes \epsilon)\mathcal{F} = w.$$

Drinfeld twists

Lemma (Main Frame)

Let $(\mathcal{A}, \Delta, \epsilon, \Phi, \mathcal{R})$ and $(\mathcal{A}, \Delta_{\mathcal{F}}, \epsilon, \Phi_{\mathcal{F}}, \mathcal{R}_{\mathcal{F}})$ be quasi-triangular quasi-bialgebras and let the conditions of the Proposition hold. We recall also: $\mathcal{F}_{1,23} := (id \otimes \Delta)\mathcal{F}$, $\mathcal{F}_{12,3} := (\Delta \otimes id)\mathcal{F}$, then $\mathcal{F}_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{F}_{12,3}$, $\mathcal{F}_{1,32}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{F}_{1,23}$. According to the Proposition we distinguish two cases:

1. *If the associators satisfy*

$$\Phi_{213}\mathcal{R}_{12} = \mathcal{R}_{12}\Phi_{123}, \qquad \Phi_{\mathcal{F}213}\mathcal{R}_{12} = \mathcal{R}_{12}\Phi_{\mathcal{F}123}, \tag{4}$$

   and $\Phi_{\mathcal{F}}$ commutes with F_{12}, then the condition (4) can be re-expressed as $\mathcal{F}_{123} := \mathcal{F}_{23}\mathcal{F}_{1,23} = \mathcal{F}_{12}\mathcal{F}^*_{12,3}$, where $\mathcal{F}^*_{12.3} = \Phi_{\mathcal{F}}\mathcal{F}_{12,3}\Phi^{-1}$. We also deduce that $\mathcal{F}^*_{21,3}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{F}^*_{12,3}$.

2. *If the associator satisfies*

$$\Phi_{132}\mathcal{R}_{23} = \mathcal{R}_{23}\Phi_{123}, \qquad \Phi_{\mathcal{F}132}\mathcal{R}_{23} = \mathcal{R}_{23}\Phi_{\mathcal{F}123}, \tag{5}$$

   and $\Phi_{\mathcal{F}}$ commutes with F_{23}, then then condition (4) is re-expressed as $\mathcal{F}_{123} := \mathcal{F}_{23}\mathcal{F}^*_{1,23} = \mathcal{F}_{12}\mathcal{F}_{12,3}$, where $\mathcal{F}^*_{1,23} = \Phi_{\mathcal{F}}^{-1}\mathcal{F}_{1,23}\Phi$. We also deduce that $\mathcal{F}^*_{1,32}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{F}^*_{1,23}$.

[*AD, Ghionis & Vlaar*] & [*AD, & Rybolowicz*]

Twists & quasi-bialgebras from braces

Drinfeld Twist

[AD & Smoktunowicz], [AD] . Let $\check{r} = \sum_{x,y\in X} e_{x,\sigma_x(y)} \otimes e_{y,\tau_y(x)}$ be the set-theoretic solution of the braid YBE, $\mathcal{P}$ is the permutation operator and $\hat{V}_k,\ V_k$ are their respective eigenvectors. Let $F^{-1} = \sum_{k=1}^{n^2} \hat{V}_k\ V_k^T$ be the similarity transformation (twist), such that $\check{r} = F^{-1}\mathcal{P}F$. Then the twist can be explicitly expressed as $F = \sum_{x\in X} e_{x,x} \otimes \mathbb{V}_x$, where we define $\mathbb{V}_x = \sum_{y\in X} e_{\sigma_x(y),y}$.

- The twist is not uniquely defined, for instance an alternative twist is $\hat{F} = \sum_{x,y\in X} e_{\tau_y(x),x} \otimes e_{y,y} = \sum_y \mathbb{W}_y \otimes e_{y,y}$.
- *Note*: The matrices $\mathbb{V}_x$ and $\mathbb{W}_x^T$ (^{T} denotes transposition), are n-dimensional representations of $\mathcal{Q}$, i.e. $q_x \mapsto \mathbb{U}_x$, where $\mathbb{U}_x \in \{\mathbb{V}_x,\ \mathbb{W}_x^T\}$,
- Both twists are still admissible in the case of non-involutive, invertible solutions.

The twisted r-matrix

Lemma

Let $\check{r}: V\otimes V \to V\otimes V$ be the set-theoretic solution of the braid equation. Let also $\mathbb{V}_x = \sum_{y\in X} e_{\sigma_x(y),y}$ and $\mathbb{W}_x = \sum_{\eta\in X} e_{\tau_x(\eta),\eta}$, $\forall x\in X$, with coproducts defined. Then $\Delta_{\mathcal{T}}(\mathbb{Y}_x)\check{r}_{\mathcal{T}} = \check{r}_{\mathcal{T}}\Delta_{\mathcal{T}}(\mathbb{Y}_x)$, where $\mathbb{Y}_x \in \{\mathbb{V}_x,\ \mathbb{W}_x\}$, $\mathcal{T}\in\{F,\ \hat{F}\}$,

$$\Delta_F(\mathbb{V}_x) = \mathbb{V}_x\otimes\mathbb{V}_x, \quad \Delta_F(\mathbb{W}_y) = \sum_{\eta,x\in X} e_{\tau_{\sigma_x(y)}(\eta),\eta}\otimes e_{\tau_{\sigma_{\tau_x(\eta)}(y)}(\sigma_\eta(x)),\sigma_\eta(x)}$$

$$\Delta_{\hat{F}}(\mathbb{V}_\eta) = \sum_{x,y\in X} e_{\sigma_{\tau_{\sigma_x(y)}(\eta)}(\tau_y(x)),\tau_y(x)}\otimes e_{\sigma_{\tau_x(\eta)}(y),y} \quad \Delta_{\hat{F}}(\mathbb{W}_y) = \mathbb{W}_y\otimes\mathbb{W}_y$$

and the twisted matrices read as

$$\check{r}_F = \sum_{x,y\in X} e_{x,\sigma_x(y)}\otimes e_{\sigma_x(y),\sigma_{\sigma_x(y)}(\tau_y(x))} \quad \& \quad \check{r}_{\hat{F}} = \sum_{x,y\in X} e_{\tau_y(x),\tau_{\tau_y(x)}(\sigma_x(y))}\otimes e_{y,\tau_y(x)}.$$

[AD & Rybolowicz]

Involutive case: $\check{r}_F = \check{r}_{\hat{F}} = \mathcal{P}$. Baxterization $\check{R}(\lambda) = \lambda\check{r} + \mathcal{P}$ and relation with Yangian [*AD, Ghionis & Vlaar*] (extended the results of [*Etingof, Shedler & Soloviev*]).

Derived solution

- Focus on $\check{r}_F = \sum_{x,y\in X} e_{x,y} \otimes e_{y,y\triangleright x}$, where $y \triangleright x = \sigma_y(\hat{\sigma}_{y^{-1}}(x))$ and $(X, \triangleright)$ is a rack, i.e. $\triangleright$ is self distributive. (Relations to racks and quandles)
- Quadratic algebra from $r_F(q\otimes 1)(1\otimes q) = (1\otimes q)(q\otimes 1)$: $q_x q_y = q_y q_{y\triangleright x}$.

Comments

- Study of the quantum group associated to the derived solution....Hopf-algebra?
- Solve eigenvalue problem for non-involutive solutions of the YBE. Challenging problem that will provide valuable info on multiplicities and hence associated symmetries.
- **Ultimate goal:** Identify spectrum and eigenstates of local Hamiltonians (quantum systems) constructed from set-theoretic solutions. Key point: *Use of Drinfeld twists!* Use info gained from the study of symmetries: *spectrum multiplicities*.
- Next challenging problem: universal R-matrix!