GROUPS WHOSE PROPER SUBGROUPS SATISFY CERTAIN PROPERTIES

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AGTA, Lecce, June 5-9, 2023

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- Joint work with Sevgi Atlihan and Martin Evans

Groups whose proper subgroups satisfy some additional property

- normal
- subnormal of bounded defect
- subnormal
- permutable
- ascendant
- serial

- abelian
- nilpotent
- soluble
- nilpotent of class c
- soluble of derived length d

Classical Results

- (Dedekind 1897, Baer 1933) Let G be a non-abelian group all of whose subgroups are normal. Then $G \cong Q_8 \times E \times A$, where E is elementary abelian 2-group and A is periodic abelian group, all elements of odd order.(Dedekind Group)

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- (Roseblade 1965) Let G be a group and let d be a natural number. If every subgroup of G is subnormal of defect at most d, then there is a function f such that G is nilpotent of class at most $f(d)$.

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- (Iwasawa 1943) Let G be a quasihamiltonian group. Then
  - G is metabelian and locally nilpotent.
  - If G is torsion-free, then G is abelian
  - If G is non-periodic, then the torsion subgroup of G is abelian.
  - If G is nonabelian, nonperiodic, then the torsion-free subgroups have rank 1.
  - If G is a nonabelian locally finite p-group, then G is either a Dedekind group or $G = A\langle b\rangle$ where A is a normal abelian subgroup of index p^m and finite exponent p^k, where $b^{-1}ab = a^{1+p^s}$ for some integer s such that $s < k \leq s + m$ for all $a \in A$. In particular G is nilpotent.

Great answers

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- A Fitting p-group with all proper subgroups soluble and hypercentral is soluble.
- (Möhres 1989) Let G be a group all of whose subgroups are subnormal. Then G is soluble.
- (Casolo 2001, Smith 2001) Let G be a torsion-free group. If all subgroups of G are subnormal, then G is nilpotent.

Interesting answers

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- This generalizes and uses a lovely theorem of D. I Zaitsev who proved this result in the soluble case
- Is an infinite locally graded group with all proper subgroups soluble necessarily soluble?
- Such groups must be hyperabelian (Franciosi, de Giovanni and Newell, 2000)

Reminders

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- Tarski monsters: finitely generated infinite simple groups all of whose proper subgroups are cyclic of prime order (Olshanskii, 1979).

All subgroups normal or abelian

- Romalis-Sesekin 1966, 68, 69: locally soluble groups with all subgroups normal or abelian.
- Kuzennyi-Semko give detailed structure.
- Brescia, Ferrara, Trombetti, 2023: Correct and add to these results.

All subgroups serial or nilpotent

- Phillips-Wilson, 1978: groups with all subgroups nilpotent or normal, locally nilpotent or normal, locally nilpotent or serial...see also Otal-Peña.

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- other authors: groups with all subgroups nilpotent or subnormal etc, serial or abelian, etc.
- see eg. Bruno, Phillips, Kurdachenko, Atlihan, Semko, Smith, MD etc.;

All subgroups subnormal or nilpotent

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3. *Let G be a locally finite group with all non-nilpotent subgroups subnormal. Then G contains a normal subgroup K of finite index in which every subgroup is subnormal.*
4. *Let G be a torsion-free locally (soluble-by-finite) group with all non-nilpotent subgroups subnormal. Then G is nilpotent.*

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3. *Let G be a locally finite group with all non-nilpotent subgroups subnormal. Then G contains a normal subgroup K of finite index in which every subgroup is subnormal.*
4. *Let G be a torsion-free locally (soluble-by-finite) group with all non-nilpotent subgroups subnormal. Then G is nilpotent.*
5. *If G is a locally graded torsion-free group and the non-nilpotent subgroups are subnormal of defect at most d, then G is nilpotent.*

Questions

- If G is an infinite locally graded group with every non-nilpotent subgroup subnormal is G soluble?
- If G is torsion-free locally graded and every non-nilpotent subgroup is subnormal is G nilpotent?
- If G is a locally graded group with all non-nilpotent subgroups subnormal does G contain a normal subgroup K of finite index in which every subgroup is subnormal?

All subgroups soluble or subnormal

Theorem (Ersoy, Tortora, Tota 2014)

Let G be a locally (soluble-by-finite) group in which every subgroup is subnormal or soluble of derived length at most d. Then

1. *G is soluble or*
2. *G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.*

Let G be a locally graded group with all subgroups subnormal of defect at most n or soluble of derived length at most d.

1. *G is soluble of derived length bounded by a function of n, d or*
2. *G is an extension of a soluble group of derived length at most d by a finite almost minimal simple group.*

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- (De Falco, de Giovanni, Musella, Schmidt 2003) G locally graded, all subgroups abelian or permutable. Then G is soluble of derived length at most 4; there is a finite normal subgroup N such that all subgroups of G/N are permutable.
- (MD, Karatas 2013) G locally graded group, all subgroups permutable or nilpotent of class at most c. Then G is soluble of derived length at most $4 + [\log_2 c]$.

Permutability Results

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- (MD, Karatas 2013) G locally graded, all subgroups permutable or soluble of derived length at most d.
  1. If G is soluble then G has derived length at most $d+3$;
  2. If G is not soluble, then G'' is finite and perfect. Also all proper subgroups of G'' are soluble of derived length at most d.

Useful facts about permutable subgroups

- (Stonehewer, 1972) If P is a permutable subgroup of a group G, then P is ascendant in G (indeed of length at most $\omega + 1$). If G is finitely generated, then P is subnormal in G.

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- If $G = H\langle g \rangle$ with H permutable in G and $\langle g \rangle$ infinite cyclic such that $\langle g \rangle \cap H = 1$, then g normalizes H.

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- If $G = H\langle g\rangle$ with H permutable in G and $\langle g\rangle$ infinite cyclic such that $\langle g\rangle \cap H = 1$, then g normalizes H.
- (De Falco, de Giovanni, Musella, Schmidt, 2003) If all subgroups of a group G containing a subgroup H are permutable and if G has an element of infinite order such that $\langle g\rangle \cap H = 1$, then H is normal in G.

Our work

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Let G be a non-periodic locally graded group and suppose every non-nilpotent subgroup of G is permutable. Then G is soluble.

General Facts and Proofs

Let G be locally graded and suppose that every non-nilpotent subgroup of G is permutable.

- If G is not soluble, then $X = G''$ is perfect.

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  $N =$ non-nilpotent proper normal subgroup of X. Then X/N is q.h.
- X is a Fitting group.
  $Y =$ product of proper normals of X. If $Y \neq X$, then Y is nilpotent and X/Y is simple so has no proper non-permutables. Asar then implies X/Y soluble
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  G not soluble implies G/X is q.h. so nilpotent. G/X periodic implies X is f.g. so there is $N \triangleleft X$ with X/N finite. But N and X/N are soluble. Thus G/X is not periodic. Finite-by-nilpt follows by Smith's results since in this case every permutable is sn.
- If G is torsion-free, then G is locally nilpotent.

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- If G is torsion-free, then G is locally nilpotent.
- If G is torsion-free, then G is soluble.

The non-periodic case

Let G be a locally graded group with all non-nilpotent subgroups permutable. If G is not periodic, then G is soluble.

- Suppose not. Then $X = G''$ is perfect, Fitting and all proper normals nilpotent.

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- If X is not periodic, let T be its torsion subgroup. T is nilpotent and X/T is torsion-free. But then X/T is soluble so X is soluble as is G, contradiction.

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- Thus X is periodic. Let H be a proper non-nilpotent subgroup of X. There is an element $g \in G$ such that $\langle g \rangle \cap H = 1$. Then H is normal in X so nilpotent. Contradiction. Thus H is nilpotent and Asar's result completes the proof.

The torsion-free case

We need in particular the following result (MD, M. Evans, H. Smith 2008)

Theorem

Let G be a locally nilpotent group with torsion subgroup T, and suppose that G/T is countable and soluble. Then there is a torsion-free, residually finite subgroup J of G such that $I_G(J) = G$.

We recall that $I_G(H) = \{x \in G | x^n \in H$ for some $n \in \mathbb{N}\}$.
Side note: This is not true in the uncountable case and may not be true in the non-soluble case.

The torsion-free case

Let G be a torsion-free locally graded group with all non-nilpotent subgroups permutable. Suppose that G is not nilpotent. We know G is soluble and locally nilpotent in this case. Let G be a counterexample of minimal derived length $d \geq 2$.

- We may assume G is metabelian.

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- Wma G is residually finite. There is a residually finite subgroup J of G such that $I_G(J) = G$. If J is nilpotent, so is G.
- Isolators imply wma G has an abelian normal subgroup L such that G/L is free abelian.

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- Then $G/L = B/L \times C/L$ where both are free abelian of infinite rank.
- One of B, C is not nilpotent (Fitting's Theorem), B say.
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- Thus G/B^n is quasihamiltonian, so has torsion-free rank 1. This is a contradiction since C/L has infinite rank.

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- Thus G/B^n is quasihamiltonian, so has torsion-free rank 1. This is a contradiction since C/L has infinite rank.
- Thus $G/L \cong \mathbb{Z} \times \cdots \times \mathbb{Z}$. Fitting implies wma G/L is infinite cyclic.

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- Let $a_1, a_2, \ldots$ freely generate A. G is nonabelian so $[a_1, x] \neq 1$ say. Then $x^{-1}a_1x = a_1^{e_1} \ldots a_r^{e_r}$.

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- Let $W_n = A^n\langle x \rangle$. $I_G(W_n) = G$ so W_n is non-nilpotent hence permutable in G.

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- Then $a_1x = x^r a_1^{d_n} b_n^n$ where $b \in \langle a_2, \dots, a_r \rangle$. Since $A \cap F = 1, r = 1$ so $x^{-1}a_1x = a_1^{d_n}b_n^n$.

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- Let $W_n = A^n\langle x \rangle$. $I_G(W_n) = G$ so W_n is non-nilpotent hence permutable in G.
- Then $a_1x = x^r a_1^{d_n} b_n^n$ where $b \in \langle a_2, \ldots, a_r \rangle$. Since $A \cap F = 1, r = 1$ so $x^{-1}a_1x = a_1^{d_n} b_n^n$.
- $n > 2$ is arbitrary and $n|e_i$ for each $i \geq 2$. Thus $e_i = 0$ for each i so $x^{-1}a_1x = a_1^{e_1}$. But $\langle a_1, x \rangle$ is nilpotent. Thus $e_1 = 1$ and we obtain the final contradiction.

Remarks on the torsion case

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- At this moment we know that if it is not, then there is a perfect Fitting p-group X with all subgroups permutable or nilpotent. We may assume the proper normal subgroups of X are nilpotent and all proper subgroups are soluble. We know a number of other facts concerning such an X.

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- But at this point we can only summon the services of Harry Potter to complete the story!

Thanks

Grazie mille!
Ho apprezzato la mia visita.
And thank you for a wonderful conference!