Trusses *vs* Rings

Tomasz Brzeziński

Swansea University & University of Białystok

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Associative algebras on affine spaces or *affgebras*

- An associative algebra is a vector space A with an associative bi-linear multiplication $\mathrm{m} : A \times A \to A$.
- The bi-linearity of m implies that multiplication distributes over the addition according to the ring distributive law.
- What an affine space with an associative bi-affine multiplication $\mathrm{m} : A \times A \to A$ is?

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  $$a + \lambda\overrightarrow{ab} \in A.$$
- Observation: we can get rid of V altogether (but then recover it up to isomorphism!).

Heaps [Prüfer ’24, Baer ’29]

Definition

A **heap** is a nonempty set A together with a ternary operation

$$[-,-,-] : A \times A \times A \to A,$$

such that for all $a_i \in A$, $i = 1, \ldots, 5$,

(a) $[[a_1, a_2, a_3], a_4, a_5] = [a_1, a_2, [a_3, a_4, a_5]]$,

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Homomorphism of heaps: a function $f : A \to B$ such that

$$f[a_1, a_2, a_3] = [f(a_1), f(a_2), f(a_3)].$$

Heaps are in ‘1-1’ correspondence with groups

- If $(A, +)$ is a (-n abelian) group, then A is a (-n abelian) heap $\mathcal{H}(A)$ with operation

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  makes A into a (-n abelian) group $\mathcal{G}(A,e)$ (with identity e and the inverse mapping $a \mapsto [e,a,e]$).
- $\mathcal{G}(A,e) \cong \mathcal{G}(A,f)$.
- $\mathcal{H} \circ \mathcal{G} = \mathrm{id}$, i.e., irrespective of e:

$$[a,b,c]_{+_e} = [a,b,c].$$

Affine spaces (cd) [BBRS]

An affine space A is a heap with an $\mathbb{F}$-action (heap of $\mathbb{F}$-modules) $(\lambda, a, b) \mapsto \lambda \triangleright_a b$, such that

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is a vector space. A vector from a to b: $\overrightarrow{ab} = [o, a, b]$.

Affine transformation

A morphism of affine spaces (A, V) to (B, W) is a function $f : A \to B$ which induces a linear transformation $\hat{f} : V \to W$ such that

$$\hat{f}\left(\overrightarrow{ab}\right) = \overrightarrow{f(a)f(b)}.$$

This is equivalent to say that f is a morphism of heaps such that

$$f(\lambda \triangleright_a b) = \lambda \triangleright_{f(a)} f(b)$$

Trusses [TB '19]

Let A be an affine space with an associative bi-affine multiplication

$$\mathrm{m} : A \times A \to A, \qquad (a, b) \longmapsto ab.$$

What kind of distributivity we get?

(i) m is a heap morphism on the left argument:

$$[a, b, c]d = [ad, bd, cd],$$

(ii) m is a heap morphism on the right argument:

$$a[b, c, d] = [ab, ac, ad].$$

An abelian heap A with an associative multiplication satisfying (i)-(ii) is called a **truss.**

Comments on trusses

- In a ring, $a0 = 0a = 0$, by the distributive laws.
- The truss distributive laws on an abelian group heap read:

  $$(a - b + c)d = ad - bd + cd, \qquad a(b - c + d) = ab - ac + ad.$$

  This **does not** imply that $a0 = 0a = 0$.
- An abelian group is a truss with any of these products:

  $$ab = a + b, \quad ab = a, \quad ab = b, \quad ab = \text{const}.$$

- Odd integers and odd fractions are trusses with the usual multiplication.

Two types of distributive laws

Ring-type: $(R, +, \cdot)$:

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Brace-type: $(R, +, \cdot)$:

$$a(b+c) = ab-a+ac, \qquad (b+c)a = ba-a+ca \ (B).$$

Consequence: $0 = 1$, hence all elements can be invertible.

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- Let $(A, [-,-,-], \cdot)$ be a truss. Assume that $0 \in A$ is such that
$$a \cdot 0 = 0 = 0 \cdot a, \qquad \text{for all } a \in A.$$
Then $(A, +_0, \cdot)$ is a ring.

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Trusses can be understood as:

- Algebras on affine $\mathbb{Z}$-spaces (compare: rings are algebras over $\mathbb{Z}$).
- Slices of rings over $\mathbb{Z}$. If $f : R \longrightarrow \mathbb{Z}$ is a surjective (non-unital) ring homomorphism, then $f^{-1}(1) \subseteq R$ is a truss [RR Andruszkiewicz, TB, B Rybołowicz '22].

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(2) $S(T;o)=R(T;o)\oplus\mathbb{Z}$ *is a ring with:*

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(4) $T\cong\{(a,1)\mid a\in T\}, \qquad a\longmapsto(a,1).$

Truss structures on $(\mathbb{Z}, [- - -]_+)$:

Theorem

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(3) Isomorphism classes of truss structures in (2) are in 1-1 correspondence with orbits of the action of

$$D_\infty = \left\{ \begin{pmatrix} 1 & 0 \\ k & \pm 1 \end{pmatrix} \mid k \in \mathbb{Z} \right\}.$$

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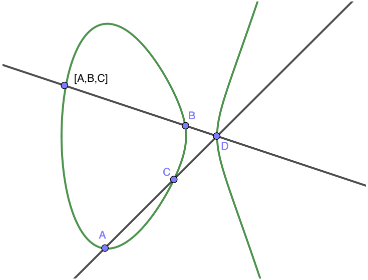
$$a \cdot (b + c) = \kappa(a, b) + \hat{\kappa}(a, c).$$
- $a \cdot [b, c, d]_+ = [a \cdot b,\ a \cdot c,\ a \cdot d]_+.$

Trusses and ellitpic curves



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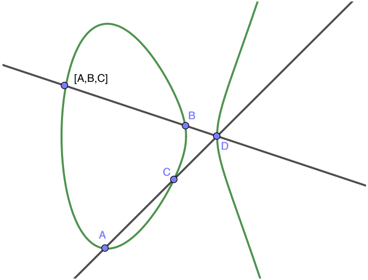

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- *Endomorphisms of $\mathcal{E}$ are endomorphisms of $H(\mathcal{E})$.*

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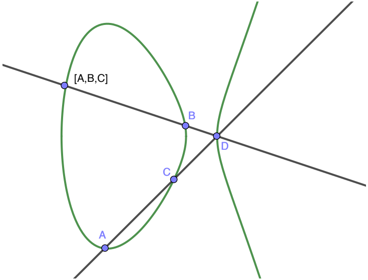

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Theorem

Let $\mathcal{E}$ be a nonsingular complex elliptic curve.

- *Endomorphisms of $\mathcal{E}$ are endomorphisms of $H(\mathcal{E})$.*
- *Endomorphisms of $\mathcal{E}$ form a truss $T(\mathcal{E})$ with product $\circ$ and*

$$[f, g, h](A) = [f(A), g(A), h(A)], \qquad \textit{for all } A \in \mathcal{E},$$

Summary

- Affine spaces are equipped with a natural ternary operation that makes them into abelian heaps.
- An affine space with a bi-affine multiplication becomes a truss (multiplication distributes over the ternary operation).
- Every ring is a truss, every brace is a truss; trusses are a bridge between rings and braces.
- All trusses can be embedded universally in rings (albeit as trusses NOT as rings).
- All trusses arise as (nonunital) extensions of $\mathbb{Z}$ by ideals.

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