On Quadratic Rational Groups

Marco Vergani

Università degli Studi di Firenze

AGTA - 8 June 2023

Introduction

1. Dual definition from “character table perspective” of semirational groups.

Introduction

1. Dual definition from "character table perspective" of semirational groups.
2. Related to "nice" characterization for central units of $\mathcal{U}(\mathbb{Z}G)$.

Introduction

1. Dual definition from "character table perspective" of semirational groups.
2. Related to "nice" characterization for central units of $\mathcal{U}(\mathbb{Z}G)$.
3. Nice bound of the spectra in case of solvable quadratic rational groups, important to study Gruenberg-Kegel graphs.

Notation

- Every group is finite.

Notation

- Every group is finite.
- $x \sim y$ denotes the conjugation in the group.

Notation

- Every group is finite.
- $x \sim y$ denotes the conjugation in the group.
- $|g|$ is the order of the element $g \in G$.
- $B_G(g) := \frac{N_G(<g>)}{C_G(<g>)}$

Notation

- Every group is finite.
- $x \sim y$ denotes the conjugation in the group.
- $|g|$ is the order of the element $g \in G$.
- $B_G(g) := \frac{N_G(<g>)}{C_G(<g>)}$
- $\mathbb{Q}_n := \mathbb{Q}(e^{2\pi i/n})$

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

Definition

An element $x \in G$ is called **semirational** iff $\exists m_x$ such that for every $(j, |x|) = 1$ then $x^j \sim x$ or $x^j \sim x^{m_x}$.

$B_G(x) \leq \mathrm{Aut}(\langle x \rangle)$

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

Definition

An element $x \in G$ is called **semirational** iff $\exists m_x$ such that for every $(j, |x|) = 1$ then $x^j \sim x$ or $x^j \sim x^{m_x}$.

$$B_G(x) \overbrace{\leq}^{2} \mathrm{Aut}(\langle x \rangle)$$

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

Definition

An element $x \in G$ is called **semirational** iff $\exists m_x$ such that for every $(j, |x|) = 1$ then $x^j \sim x$ or $x^j \sim x^{m_x}$.A group is called **semirational** iff every element is semirational. A group is called **r-semirational** if for any $x \in G$ $m_x = r$.

$$B_G(x) \overset{2}{\leq} \mathrm{Aut}(\langle x \rangle)$$

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

Definition

An element $x \in G$ is called **semirational** iff $\exists m_x$ such that for every $(j, |x|) = 1$ then $x^j \sim x$ or $x^j \sim x^{m_x}$.A group is called **semirational** iff every element is semirational. A group is called **r-semirational** if for any $x \in G$ $m_x = r$.In particular a group is called **inverse-semirational** if is $-1-$semirational.

$$\overbrace{B_G(x) \leq \mathrm{Aut}}^{2}(\langle x \rangle)$$

Examples

- Rational groups, in particular S_n.

Examples

- Rational groups, in particular S_n.
- $\mathbb{M}$ is inverse semirational.

Examples

- Rational groups, in particular S_n.
- $\mathbb{M}$ is inverse semirational.
- A_n is quadratic rational and semirational, in general not inverse-semirational.

Examples

- Rational groups, in particular S_n.
- $\mathbb{M}$ is inverse semirational.
- A_n is quadratic rational and semirational, in general not inverse-semirational.
- D_{10} is ± 3-semirational but not inverse semirational.

Examples

- Rational groups, in particular S_n.
- $\mathbb{M}$ is inverse semirational.
- A_n is quadratic rational and semirational, in general not inverse-semirational.
- D_{10} is ± 3-semirational but not inverse semirational.
- `SmallGroup(32, 42)` is quadratic rational but not semirational.

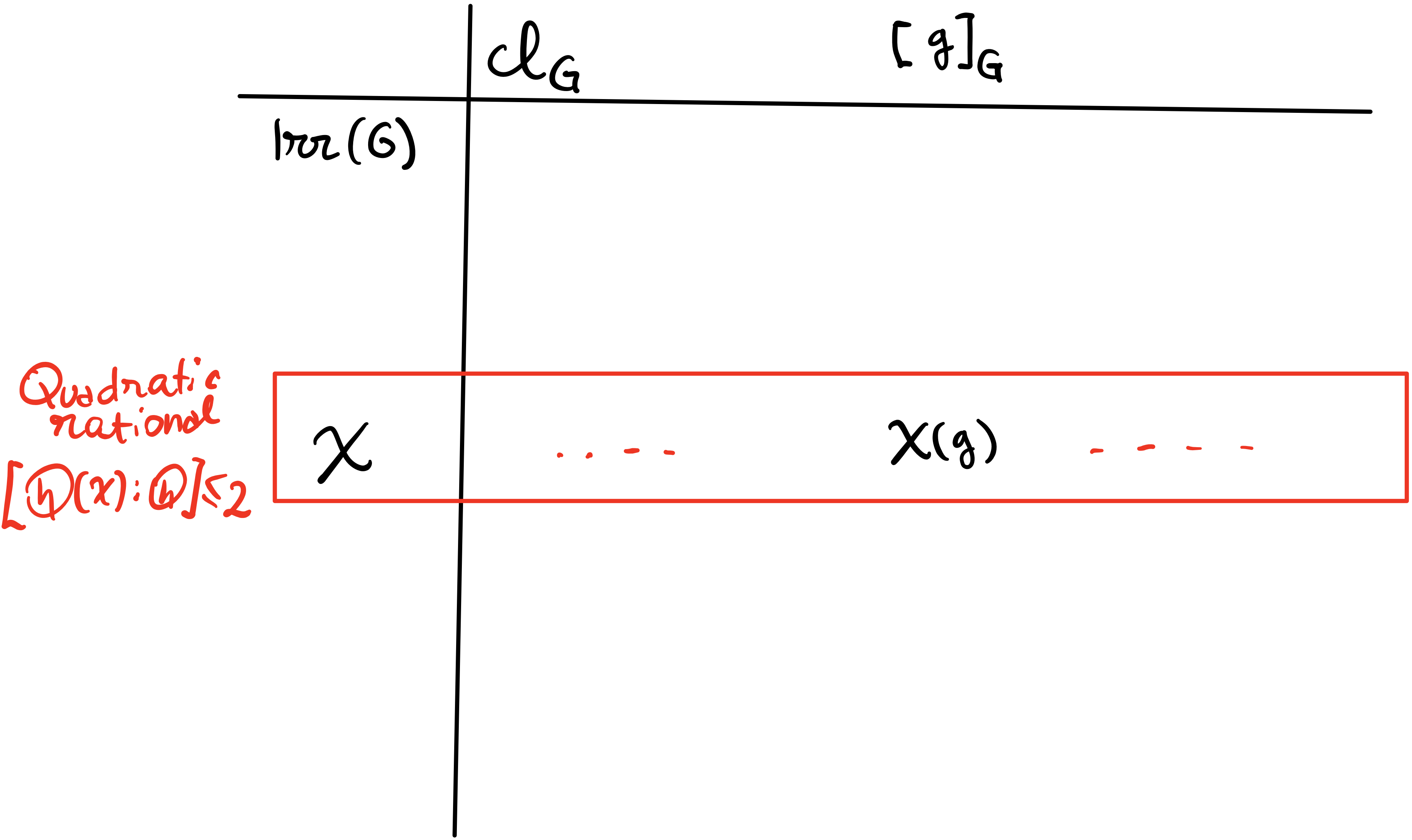
Examples

- Rational groups, in particular S_n.
- $\mathbb{M}$ is inverse semirational.
- A_n is quadratic rational and semirational, in general not inverse-semirational.
- D_{10} is ± 3-semirational but not inverse semirational.
- `SmallGroup(32, 42)` is quadratic rational but not semirational.
- `SmallGroup(32, 9)` is semirational but not quadratic rational.

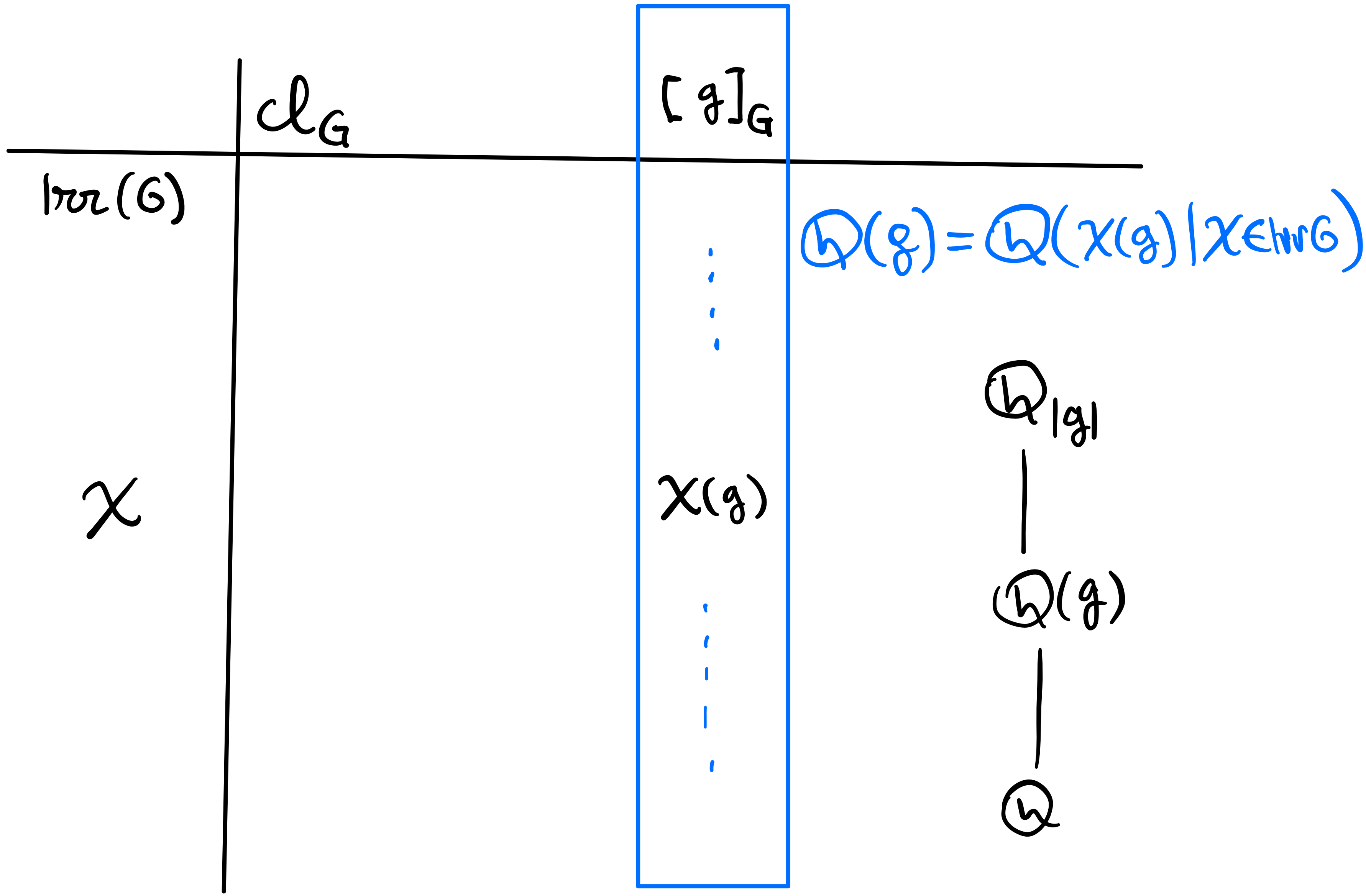
Character Table Duality

	cl_G	$[g]_G$
$Irr(G)$		
χ		$\chi(g)$

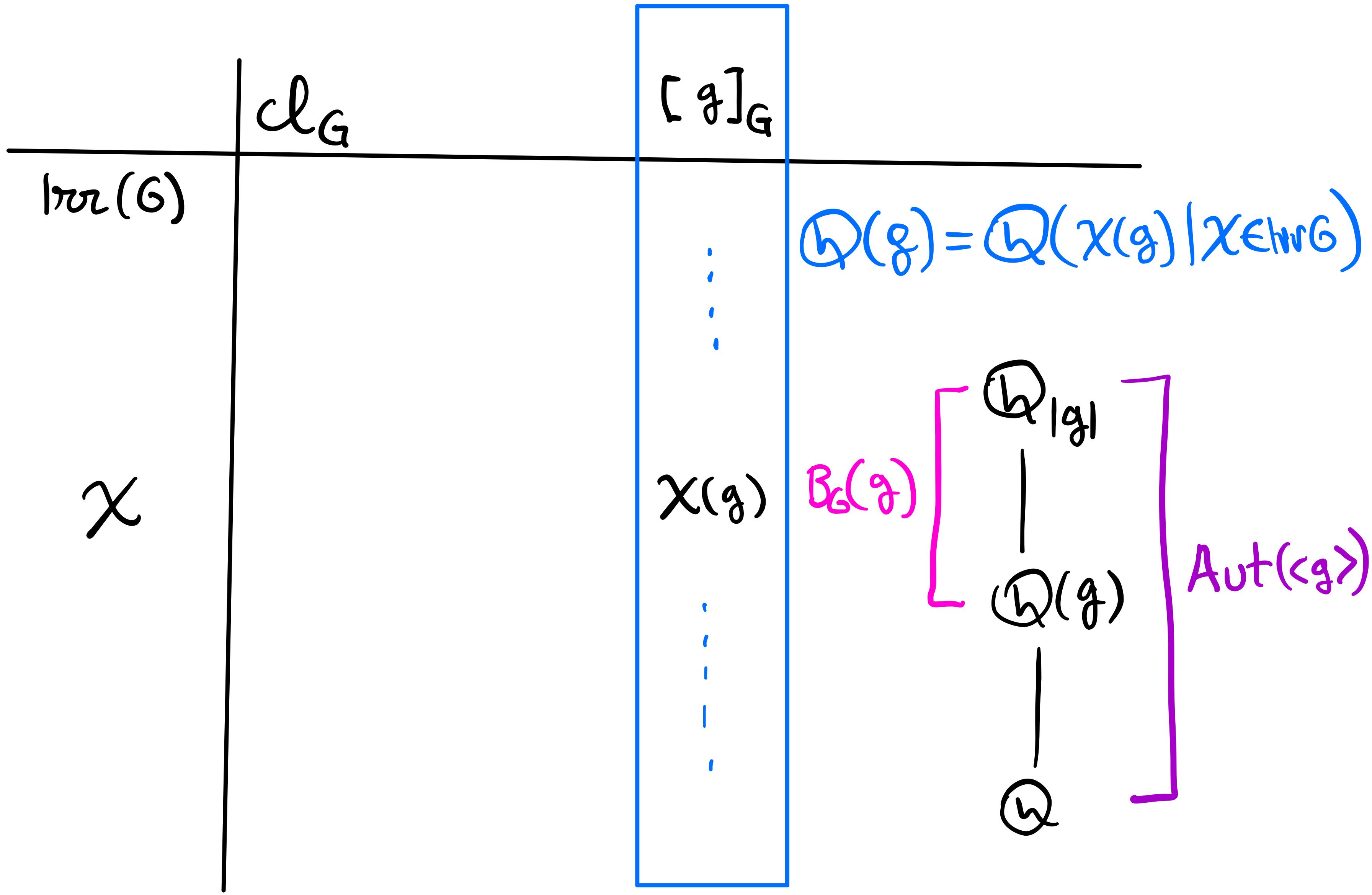
Character Table Duality



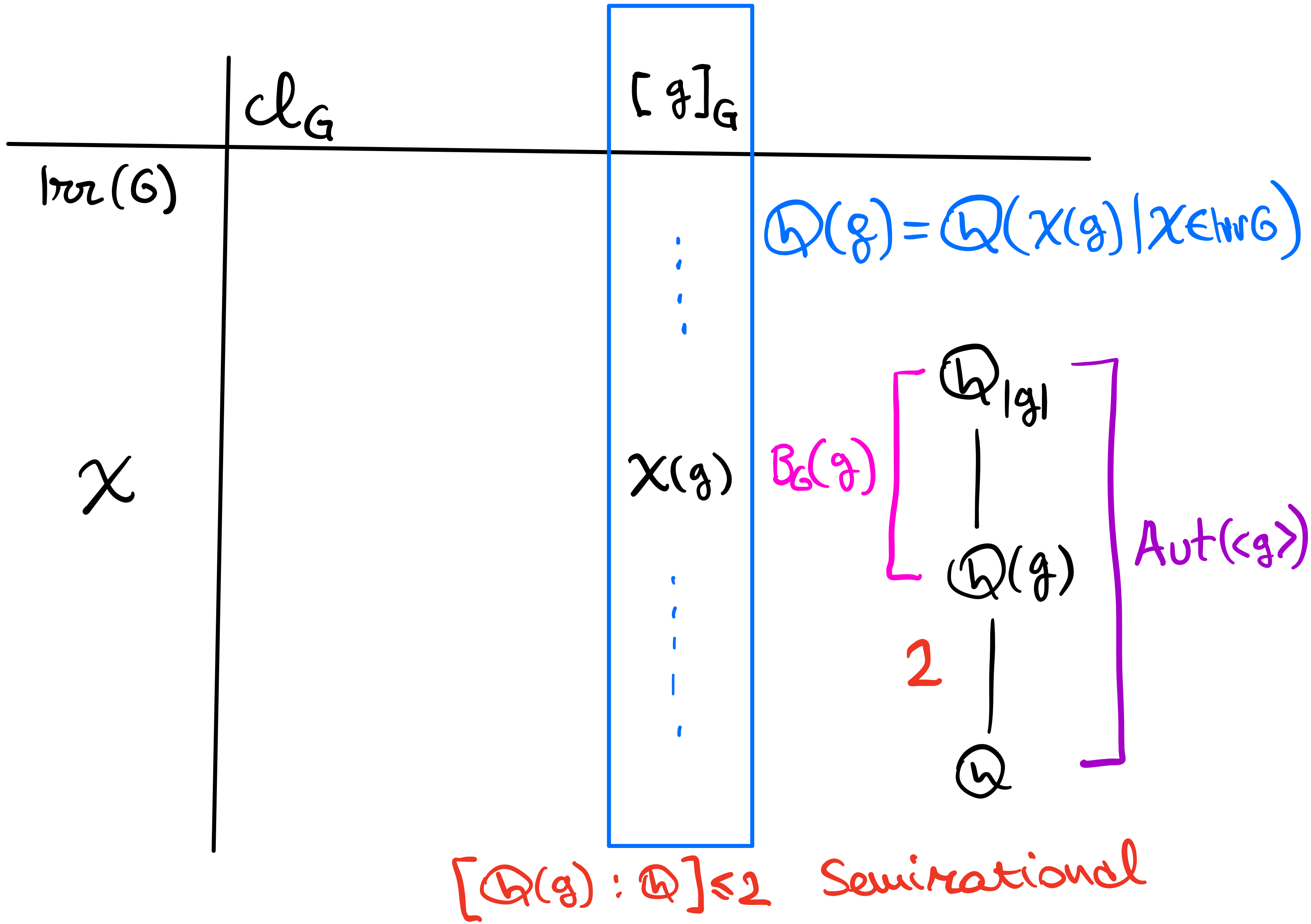
Character Table Duality



Character Table Duality



Character Table Duality



First Proprieties

Lemma

Let G be a finite group.

First Proprieties

Lemma

Let G be a finite group.

1. *If G is quadratic rational (or semirational) and N is a normal subgroup of G, then G/N is quadratic rational (or semirational).*

First Proprieties

Lemma

Let G be a finite group.

1. *If G is quadratic rational (or semirational) and N is a normal subgroup of G, then G/N is quadratic rational (or semirational).*
2. *If G is abelian, then G is quadratic rational (or semirational) if and only if the orders of the elements of G belong to $\{1, 2, 3, 4, 6\}$.*

First Proprieties

Lemma

Let G be a finite group.

1. *If G is quadratic rational (or semirational) and N is a normal subgroup of G, then G/N is quadratic rational (or semirational).*
2. *If G is abelian, then G is quadratic rational (or semirational) if and only if the orders of the elements of G belong to $\{1, 2, 3, 4, 6\}$.*
3. *If G quadratic rational, then the group of central units of $\mathbb{Z}G$ is finitely generated and the number of generator is exactly the number of irreducible character with real quadratic extension.*

Cut groups

In general we have the inclusion

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \geq \pm\mathcal{Z}(G)$$

but there is a family of groups that satisfies the following equality:

Cut groups

In general we have the inclusion

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) \geq \pm\mathcal{Z}(G)$$

but there is a family of groups that satisfies the following equality:

Definition

A finite group G is called **cut** (central units trivial) iff

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm\mathcal{Z}(G)$$

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.
2. G is inverse semirational.

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.
2. G is inverse semirational.
3. For any $x \in G$ then either $|B_G(x)| = \varphi(|x|)$

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.
2. G is inverse semirational.
3. For any $x \in G$ then either $|B_G(x)| = \varphi(|x|)$or $|B_G(x)| = \varphi(|x|)/2$ and $x \not\sim x^{-1}$.

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.
2. G is inverse semirational.
3. For any $x \in G$ then either $|B_G(x)| = \varphi(|x|)$or $|B_G(x)| = \varphi(|x|)/2$ and $x \not\sim x^{-1}$.
4. If $\mathbb{Q}G \cong \bigoplus_{k=1}^{m} M_{n_k}(D_k)$ is the Wedderburn decomposition where $m, n_k \in \mathbb{Z}_{\geq 1}$ and D_k rational division algebras for each k, then

$$\mathcal{Z}(D_k) \cong \mathbb{Q}(\sqrt{-d})$$

for some $d \in \mathbb{Z}_{\geq 0}$ square free.

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.
2. G is inverse semirational.
3. For any $x \in G$ then either $|B_G(x)| = \varphi(|x|)$or $|B_G(x)| = \varphi(|x|)/2$ and $x \not\sim x^{-1}$.
4. If $\mathbb{Q}G \cong \bigoplus_{k=1}^{m} M_{n_k}(D_k)$ is the Wedderburn decomposition where $m, n_k \in \mathbb{Z}_{\geq 1}$ and D_k rational division algebras for each k, then

$$\mathcal{Z}(D_k) \cong \mathbb{Q}(\sqrt{-d})$$

   for some $d \in \mathbb{Z}_{\geq 0}$ square free.
5. For each $\chi \in \mathrm{Irr}(G)$, the field of values of χ is $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{Z}_{\geq 1}$, i.e. has field of values equal to $\mathbb{Q}$ or an immaginary quadratic extension of $\mathbb{Q}$.

Cut equivalences

Proposition (Bächle, 2017)

The following are equivalent.

1. G is **cut**.
2. G is inverse semirational.
3. For any $x \in G$ then either $|B_G(x)| = \varphi(|x|)$or $|B_G(x)| = \varphi(|x|)/2$ and $x \not\sim x^{-1}$.
4. If $\mathbb{Q}G \cong \bigoplus_{k=1}^{m} M_{n_k}(D_k)$ is the Wedderburn decomposition where $m, n_k \in \mathbb{Z}_{\geq 1}$ and D_k rational division algebras for each k, then
$$\mathcal{Z}(D_k) \cong \mathbb{Q}(\sqrt{-d})$$
for some $d \in \mathbb{Z}_{\geq 0}$ square free.
5. For each $\chi \in \mathrm{Irr}(G)$, the field of values of χ is $\mathbb{Q}(\chi) = \mathbb{Q}(\sqrt{-d})$ for some $d \in \mathbb{Z}_{\geq 1}$, i.e. has field of values equal to $\mathbb{Q}$ or an immaginary quadratic extension of $\mathbb{Q}$. $\mathbb{Q}(\chi) \cap \mathbb{R} = \mathbb{Q}$

More symmetries of inverse-semirationals

Theorem (Bächle, Caicedo, Jespers, Maheshwary, 2021)

Let G be a inverse-semirational group of exponent dividing n. Then the natural actions of $\mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ on the conjugacy classes and on the irreducible characters of G are **permutation isomorphic**. In particular, the number of rational irreducible characters of G is equal to the number of rational conjugacy classes of G.

Solvable case

Since we are interested in studing the Gruenberg-Kegel graph of those groups, would be nice to have a bound over the prime spectra mainly in the solvable case, let us denote $\pi(G) := \{p | p \mid |G|\}$

Solvable case

Since we are interested in studing the Gruenberg-Kegel graph of those groups, would be nice to have a bound over the prime spectra mainly in the solvable case, let us denote $\pi(G) := \{p | p \mid |G|\}$

Theorem (Tent, 2012)

Let G be a solvable quadratic rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}.$$

Solvable case

Since we are interested in studing the Gruenberg-Kegel graph of those groups, would be nice to have a bound over the prime spectra mainly in the solvable case, let us denote $\pi(G) := \{p | p \mid |G|\}$

Theorem (Tent, 2012)

Let G be a solvable quadratic rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}.$$

Theorem (Chilligan, Dolfi 2010-Bächle 2017)

Let G be a solvable semirational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}.$$

Solvable case

Since we are interested in studing the Gruenberg-Kegel graph of those groups, would be nice to have a bound over the prime spectra mainly in the solvable case, let us denote $\pi(G) := \{p | p \mid |G|\}$

Theorem (Tent, 2012)

Let G be a solvable quadratic rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}.$$

Theorem (Chilligan, Dolfi 2010-Bächle 2017)

Let G be a solvable semirational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}.$$

If the group is inverse-semirational then

$$\pi(G) \subseteq \{2, 3, 5, 7\}.$$

Solvable case

Since we are interested in studing the Gruenberg-Kegel graph of those groups, would be nice to have a bound over the prime spectra mainly in the solvable case, let us denote $\pi(G) := \{p | p \mid |G|\}$

Theorem (Tent, 2012)

Let G be a solvable quadratic rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}.$$

Theorem (Chilligan, Dolfi 2010-Bächle 2017)

Let G be a solvable semirational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7, 13, 17\}.$$

?

If the group is inverse-semirational then

$$\pi(G) \subseteq \{2, 3, 5, 7\}.$$

Quasi-rational group

We have seen that inverse-semirational groups have a lot of proprieties regarding the symmetries of their character table.

Quasi-rational group

We have seen that inverse-semirational groups have a lot of proprieties regarding the symmetries of their character table.
We worked on constructing a meaningful generalization that preserves a lot of those symmetries.

Quasi-rational group

We have seen that inverse-semirational groups have a lot of proprieties regarding the symmetries of their character table.
We worked on constructing a meaningful generalization that preserves a lot of those symmetries.

Definition

A group is called **quasi-rational** if there exists $r \in \mathbb{Z}$ such that $(r, exp(G)) = 1$ and G is r-semirational.

Quasi-rational group

We have seen that inverse-semirational groups have a lot of proprieties regarding the symmetries of their character table.
We worked on constructing a meaningful generalization that preserves a lot of those symmetries.

Definition

A group is called **quasi-rational** if there exists $r \in \mathbb{Z}$ such that $(r, exp(G)) = 1$ and G is r-semirational.

Theorem (MV)

Let G be a solvable quasi-rational group. Then

$$\pi(G) \subseteq \{2, 3, 5, 7\}.$$

.

Equivalences of r-semirational groups

Proposition (MV)

Let G be a group with exponent n,

Equivalences of r-semirational groups

Proposition (MV)

Let G be a group with exponent n, $\mathbb{F}$ be a subfield of $\mathbb{Q}_n$ fixed by the cyclic subgroup generated by $\sigma_r \in \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ such that $(r, n) = 1$ and $\sigma_r(\zeta_n) = \zeta_n^r$. Then the following are equivalent:

Equivalences of r-semirational groups

Proposition (MV)

Let G be a group with exponent n, $\mathbb{F}$ be a subfield of $\mathbb{Q}_n$ fixed by the cyclic subgroup generated by $\sigma_r \in \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ such that $(r, n) = 1$ and $\sigma_r(\zeta_n) = \zeta_n^r$. Then the following are equivalent:

1. G is r-semirational.

Equivalences of r-semirational groups

Proposition (MV)

Let G be a group with exponent n, $\mathbb{F}$ be a subfield of $\mathbb{Q}_n$ fixed by the cyclic subgroup generated by $\sigma_r \in \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ such that $(r, n) = 1$ and $\sigma_r(\zeta_n) = \zeta_n^r$. Then the following are equivalent:

1. G is r-semirational.
2. $\forall \chi \in \mathrm{Irr}(G)$ $\mathbb{Q}(\chi)$ is quadratic or rational and $\mathbb{Q}(\chi) \cap \mathbb{F} = \mathbb{Q}$.

Equivalences of r-semirational groups

Proposition (MV)

Let G be a group with exponent n, $\mathbb{F}$ be a subfield of $\mathbb{Q}_n$ fixed by the cyclic subgroup generated by $\sigma_r \in \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ such that $(r, n) = 1$ and $\sigma_r(\zeta_n) = \zeta_n^r$. Then the following are equivalent:

1. G is r-semirational.
2. $\forall \chi \in \mathrm{Irr}(G)$ $\mathbb{Q}(\chi)$ is quadratic or rational and $\mathbb{Q}(\chi) \cap \mathbb{F} = \mathbb{Q}$.

Theorem (MV)

Let G be a quasi-rational group of exponent dividing n. Then the natural actions of $\mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ on the conjugacy classes and on the irreducible characters of G are **permutation isomorphic**. In particular, the number of rational irreducible characters of G is equal to the number of rational conjugacy classes of G.

Which r makes a group r-semirational

We have seen that the same group can have different integer r such that G is r-semirational.

Which r makes a group r-semirational

We have seen that the same group can have different integer r such that G is r-semirational.

Definition

Let G be a quasi-rational group and $n = exp(G)$ then we call:

$$R_G := \{r \in \mathcal{U}_n | G \text{ is } r - semirational\}.$$

Which r makes a group r-semirational

We have seen that the same group can have different integer r such that G is r-semirational.

Definition

Let G be a quasi-rational group and $n = exp(G)$ then we call:

$$R_G := \{r \in \mathcal{U}_n | G \text{ is } r - semirational\}.$$

We can observe that, fixed the group G, R_G is the lateral of the group:

$$H_G = \{r \in (\mathbb{Z}/n\mathbb{Z})^{\times} | g^r \sim g \; \forall g \in G\} \cong \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q}(G))$$

.

Some questions

What kind of R_G can appear?

Some questions

What kind of R_G can appear?
In particular, can all possible laterals of "compatible" groups H_G appear?

Some questions

What kind of R_G can appear?
In particular, can all possible laterals of "compatible" groups H_G appear?

Table: Possible R_G for quasi-rational 2-groups with exponent at least 8

$\{-1, 3\}$	$\{-1, -3\}$	$\{3, -3\}$	$\{-1\}$	$\{3\}$	$\{-3\}$
$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^{-3}$	$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^3$	$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x\rangle_2$ $a^x = a^3$ $b^x = a^4 b^{-1}$ $a^y = a^5 b^2$ $b^y = b^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x, y\rangle_2$ $a^x = a^{-1}$ $b^x = a^4 b^{-1}$ $a^y = a^5 b^2$ $b^y = b^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x, y\rangle_2$ $a^x = a^{-1}$ $b^x = a^4 b^{-1}$ $a^y = a^5 b^2$ $b^y = a^4 b$

$\{2,3\}-$groups

$\{\pm 5, \pm 7\}$	$\{\pm 7, \pm 11\}$	$\{\pm 5, \pm 11\}$
$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{-1}$ $a^y = a^{-11}$ SmallGroup(96,115)	$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{-1}$ $a^y = a^{-5}$ SmallGroup(96,121)	$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{-1}$ $a^y = a^{7}$ SmallGroup(96,117)
$\{-1,-7,5,11\}$	$\{-1,-11,5,7\}$	$\{-1,-5,7,11\}$
$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{-11}$ $a^y = a^{-5}$ SmallGroup(96,183)	$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{-5}$ $a^y = a^{11}$ SmallGroup(96,120)	$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{5}$ $a^y = a^{-11}$ SmallGroup(96,113)
	$\{-1,-11,-5,-7\}$	
	$\langle a\rangle_{24} : \langle x,y\rangle_2$ $a^x = a^{5}$ $a^y = a^{11}$ SmallGroup(96,118)	
$\{-1,11\}$	$\{7,-5\}$	$\{7,11\}$
SmallGroup(192,95)	SmallGroup(192,305)	SmallGroup(192,412)
$\{5,7\}$	$\{-1,-7\}$	$\{\pm 7\}$
SmallGroup(192,414)	SmallGroup(192,713)	SmallGroup(192,415)
$\{-1,7\}$	$\{-7,-5\}$	$\{5,-7\}$
SmallGroup(192,418)	SmallGroup(192,435)	SmallGroup(192,623)
$\{-1,-5\}$	$\{\pm 5\}$	$\{11,-5\}$
SmallGroup(192,440)	SmallGroup(192,949)	SmallGroup(192,438)
$\{-1,5\}$	$\{5,11\}$	$\{11,-7\}$
SmallGroup(192,1396)	SmallGroup(192,632)	SmallGroup(192,726)
$\{7\}$	$\{-5\}$	$\{-1\}$
SmallGroup(192,424)	SmallGroup(192,445)	SmallGroup(192,634)
$\{5\}$	$\{11\}$	$\{-11\}$
SmallGroup(192,595)	SmallGroup(192,631)	?

That's all Folks!

Thank You!