Studying two-sided skew braces

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Rings and monoids

Observation

Let R be a ring, then $(R, \circ)$ with $r \circ s = r + rs + s$ for all $r, s \in R$ is a monoid with identity 0.

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Let R be a ring, then $(R, \circ)$ with $r \circ s = r + rs + s$ for all $r, s \in R$ is a monoid with identity 0.

- For R a nil ring: $r \circ \sum_{i=1}^{\infty}(-1)r^i = 0$
- For $\mathbb{F}_2$, we find

	0	1
0	0	1
1	1	1

Jacobson radical rings

Proposition

$(R, \circ)$ is a group if and only if R coincides with its Jacobson radical.

Definition

A ring that coincides with its Jacobson radical is called a *Jacobson radical ring*

In this case, for all $r, s, t \in R$:

$$r \circ (s + t) = r \circ s - r + r \circ t, \tag{1}$$

$$(r + s) \circ t = r \circ t - t + s \circ t. \tag{2}$$

Braces

Observation Rump: Jacobson radical rings give set-theoretic solutions of the YBE.

Definition ([Rum07])

A *(left) brace* is a triple $(A, +, \circ)$ such that

1. $(A, +)$ is an abelian group (called the *additive group*),
2. $(A, \circ)$ is a group (called the *multiplicative group*),
3. $a \circ (b + c) = a \circ b - a + a \circ c$ for all $a, b, c \in A$.

$(A, +, \circ)$ is a *two-sided brace* if also for all $a, b, c \in A$:

$$(a + b) \circ c = a \circ c - c + b \circ c.$$

Skew braces

Observation Rump: Jacobson radical rings give set-theoretic solutions of the YBE.

Definition ([GV17])

A *skew (left) brace* is a triple $(A, +, \circ)$ such that

1. $(A, +)$ is a group (called the *additive group*),
2. $(A, \circ)$ is a group (called the *multiplicative group*),
3. $a \circ (b + c) = a \circ b - a + a \circ c$ for all $a, b, c \in A$.

$(A, +, \circ)$ is a *two-sided skew brace* if also for all $a, b, c \in A$:

$$(a + b) \circ c = a \circ c - c + b \circ c.$$

Two-sided braces

Bijective correspondence:

$$
\begin{array}{ccc}
\{\text{two-sided braces}\} & \longleftrightarrow & \{\text{Jacobson radical rings}\}, \\
(A, +, \circ) & \mapsto & (A, +, *), \\
(R, +, \circ) & \leftarrow\!\shortmid & (R, +, \cdot)
\end{array}
$$

where

$$a * b = -a + a \circ b - b$$

for all $a, b \in A$ and

$$r \circ s = r + st + s$$

for all $r, s \in R$.

Examples

Example (Trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ, \circ)$ is a two-sided skew brace.

Example (Almost trivial skew brace)

Let $(A, \circ)$ be a group, then $(A, \circ_{\mathrm{op}}, \circ)$ is a two-sided skew brace.
Here $a \circ_{\mathrm{op}} b = b \circ a$.

Some ideals

Example

Let A^2 be the subgroup of $(A, +)$ generated by

$$\{a * b \mid a, b \in A\},$$

then A^2 is an ideal of A and A/A^2 is a trivial skew brace.

Example

Let A^2_{op} be the subgroup of $(A, +)$ generated by

$$\{a *_{\mathrm{op}} b \mid a, b \in A\},$$

then A^2_{op} is an ideal of A and A/A^2_{op} is an almost trivial skew brace. Here $a *_{\mathrm{op}} b = -b + a \circ b - a$.

Motivation

- Jacobson radical rings: well-studied
- Two-sided skew braces: only general results were obtained by Nasybullov [Nas19]

Question

How restrictive is the condition of two-sidedness for skew braces?

Two-sided skew braces

Let A be a two-sided skew brace. Then for all $a, b, c, d \in A$,

$$\begin{aligned}(a+b)\circ(c+d) &= (a+b)\circ c-(a+b)+(a+b)\circ d\\ &= a\circ c-c+b\circ c-b-a+a\circ d-d+b\circ d\\ &= a\circ c+b*_{\mathrm{op}}c+a*d+b\circ d\end{aligned}$$

but also

$$\begin{aligned}(a+b)\circ(c+d) &= a\circ(c+d)-(c+d)+b\circ(c+d)\\ &= a\circ c-a+a\circ d-d-c+b\circ c-b+b\circ d\\ &= a\circ c+a*d+b*_{\mathrm{op}}c+b\circ d.\end{aligned}$$

so $b*_{\mathrm{op}}c+a*d=a*d+b*_{\mathrm{op}}c$.

Two-sided skew braces

Proposition ([T22])

Let A be a two-sided skew brace, then A^2 centralizes A^2_{op} in $(A, +)$.

Theorem ([T22])

Let A be a two-sided skew brace, then $(A^2 \cap A^2_{\mathrm{op}}, +)$ is abelian, so $A^2 \cap A^2_{\mathrm{op}}$ is a two-sided brace.

Natural question: what can we say about $A/(A^2 \cap A^2_{\mathrm{op}})$?

Weakly trivial skew braces

Definition

A skew brace A is called *weakly trivial* if $A^2 \cap A^2_{\mathrm{op}} = 0$.

Lemma

If A be a skew brace, then $A/(A^2 \cap A^2_{\mathrm{op}})$ is weakly trivial.

Characterizing weakly trivial skew braces

Proposition ([T22])

Let $(G, \circ)$, $(H, \circ)$ *be groups. There is a bijective correspondence between weakly trivial skew braces* A *such that* $A/A^2 \cong (G, \circ, \circ)$ *and* $A/A^2_{\mathrm{op}} \cong (H, \circ_{\mathrm{op}}, \circ)$ *and normal subgroups of* $(G, \circ) \times (H, \circ)$ *such that the projections onto a single component are surjective.*

Using Goursat's lemma a complete characterization can be given.

Two-sided skew braces revisited

Recall:

1. If A is a two-sided skew brace, then $A^2 \cap A^2_{\mathrm{op}}$ is a two-sided brace.
2. If A be a skew brace, then $A/(A^2 \cap A^2_{\mathrm{op}})$ is weakly trivial.

Two-sided skew braces revisited

Recall:

1. If A is a two-sided skew brace, then $A^2 \cap A^2_{\mathrm{op}}$ is a two-sided brace.
2. If A be a skew brace, then $A/(A^2 \cap A^2_{\mathrm{op}})$ is weakly trivial.

Theorem ([T22])

Every two-sided skew brace is an extensions of a weakly trivial skew brace by a two-sided brace.

$$0 \longrightarrow A^2 \cap A^2_{\mathrm{op}} \longrightarrow A \longrightarrow A/(A^2 \cap A^2_{\mathrm{op}}) \longrightarrow 0$$

Consequences: connections $(A, +)$ and $(A, \circ)$

Theorem ([Nas19, T22])

Let A be a two-sided skew brace and $(A, \circ)$ solvable of degree n, then $(A, +)$ is solvable of degree at most $n + 1$.

Consequences: connections $(A, +)$ and $(A, \circ)$

Theorem ([Nas19, T22])

Let A be a two-sided skew brace and $(A, \circ)$ solvable of degree n, then $(A, +)$ is solvable of degree at most $n + 1$.

For finite skew braces the implication

$$(A, \circ) \text{ nilpotent} \implies (A, +) \text{ solvable}$$

holds, but

$$(A, \circ) \text{ solvable} \implies (A, +) \text{ solvable}$$

does not hold!

Future research

Question

The correspondence between Jacobson radical rings and two-sided braces can be extended to a correspondence between construction subgroups of near-rings and skew braces. How can we recognize two-sided skew braces here?

Future research

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The correspondence between Jacobson radical rings and two-sided braces can be extended to a correspondence between construction subgroups of near-rings and skew braces. How can we recognize two-sided skew braces here?

It is in general not true that two-sided skew braces correspond to right-distributive near-rings.

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