Outer automorphisms of group von Neumann algebras

D. Osin

Vanderbilt University

John von Neumann, 1903–1957

Introduced rings of operators (now known as *von Neumann algebras*) in the 1930-1940s.

Motivated by applications to

- representation theory of infinite groups
- dynamical systems
- mathematical foundations of quantum mechanics

Recall that a **-algebra* (over $\mathbb{C}$) is a unital associative algebra over $\mathbb{C}$ endowed with a unary operation * such that

$$1^* = 1, \quad (x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*.$$

Examples.

1. $\mathbb{C}$ is a *-algebra with respect to complex conjugation.

Recall that a **-algebra* (over $\mathbb{C}$) is a unital associative algebra over $\mathbb{C}$ endowed with a unary operation * such that

$$1^* = 1, \quad (x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*.$$

Examples.

1. $\mathbb{C}$ is a *-algebra with respect to complex conjugation.
2. $M_n(\mathbb{C})$ is a *-algebra, where * means conjugate transpose.

Recall that a **-algebra* (over $\mathbb{C}$) is a unital associative algebra over $\mathbb{C}$ endowed with a unary operation * such that

$$1^* = 1, \quad (x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*.$$

Examples.

1. $\mathbb{C}$ is a *-algebra with respect to complex conjugation.
2. $M_n(\mathbb{C})$ is a *-algebra, where * means conjugate transpose.
3. More generally, suppose that $\mathcal{H}$ is a Hilbert space. Then

$$B(\mathcal{H}) = \{\text{bounded linear operators on } \mathcal{H}\}$$

is a *-algebra, where, for every $A \in B(\mathcal{H})$, A^* is the adjoint operator of A.

Recall that a **-algebra* (over $\mathbb{C}$) is a unital associative algebra over $\mathbb{C}$ endowed with a unary operation * such that

$$1^* = 1, \quad (x^*)^* = x, \quad (x+y)^* = x^* + y^*, \quad (\lambda x)^* = \overline{\lambda} x^*, \quad (xy)^* = y^* x^*.$$

Examples.

1. $\mathbb{C}$ is a *-algebra with respect to complex conjugation.
2. $M_n(\mathbb{C})$ is a *-algebra, where * means conjugate transpose.
3. More generally, suppose that $\mathcal{H}$ is a Hilbert space. Then

$$B(\mathcal{H}) = \{\text{bounded linear operators on } \mathcal{H}\}$$

   is a *-algebra, where, for every $A \in B(\mathcal{H})$, A^* is the adjoint operator of A.
4. For every group G, the group ring $\mathbb{C}G$ is a *-algebra:

$$\left(\sum_{g\in G} c_g g\right)^* = \sum_{g\in G} \overline{c}_g g^{-1}.$$

Von Neumann algebras

Definition

A *von Neumann algebra* on a Hilbert space $\mathcal{H}$ is a unital *-subalgebra of $B(\mathcal{H})$ closed in the topology of pointwise convergence.

Definition

A *von Neumann algebra* on a Hilbert space $\mathcal{H}$ is a unital *-subalgebra of $B(\mathcal{H})$ closed in the topology of pointwise convergence.

Examples:

1. $B(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. In particular, $M_n(\mathbb{C})$.

Definition

A *von Neumann algebra* on a Hilbert space $\mathcal{H}$ is a unital *-subalgebra of $B(\mathcal{H})$ closed in the topology of pointwise convergence.

Examples:

1. $B(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. In particular, $M_n(\mathbb{C})$.
2. $L^\infty(\mathbb{R})$ (acting on $\mathcal{H} = L^2(\mathbb{R})$ by pointwise multiplication).

Definition

A *von Neumann algebra* on a Hilbert space $\mathcal{H}$ is a unital *-subalgebra of $B(\mathcal{H})$ closed in the topology of pointwise convergence.

Examples:

1. $B(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. In particular, $M_n(\mathbb{C})$.
2. $L^\infty(\mathbb{R})$ (acting on $\mathcal{H} = L^2(\mathbb{R})$ by pointwise multiplication).

For $S \subseteq B(\mathcal{H})$, the *commutant* of S is

$$S' = \{b \in B(\mathcal{H}) \mid bs = sb \,\forall\, s \in S\}.$$

Definition

A *von Neumann algebra* on a Hilbert space $\mathcal{H}$ is a unital *-subalgebra of $B(\mathcal{H})$ closed in the topology of pointwise convergence.

Examples:

1. $B(\mathcal{H})$ for every Hilbert space $\mathcal{H}$. In particular, $M_n(\mathbb{C})$.
2. $L^\infty(\mathbb{R})$ (acting on $\mathcal{H} = L^2(\mathbb{R})$ by pointwise multiplication).

For $S \subseteq B(\mathcal{H})$, the *commutant* of S is

$$S' = \{b \in B(\mathcal{H}) \mid bs = sb \,\forall\, s \in S\}.$$

Theorem (von Neumann's double commutant theorem)

A unital *-subalgebra A of $B(\mathcal{H})$ is a von Neumann algebra iff $A'' = A$.

Factors

The *center* of a von Neumann algebra M is defined by

$$Z(M) = \{z \in M \mid zx = xz \; \forall \, x \in M\}.$$

Factors

The *center* of a von Neumann algebra M is defined by

$$Z(M) = \{z \in M \mid zx = xz \ \forall\, x \in M\}.$$

Definition

A von Neumann algebra M is a *factor* if its center is trivial, i.e., $Z(M) = \mathbb{C}1_M$.

Factors

The *center* of a von Neumann algebra M is defined by

$$Z(M) = \{z \in M \mid zx = xz \,\forall\, x \in M\}.$$

Definition

A von Neumann algebra M is a *factor* if its center is trivial, i.e., $Z(M) = \mathbb{C}1_M$.

Example:

- $B(\mathcal{H})$ is a factor.

Factors

The *center* of a von Neumann algebra M is defined by

$$Z(M) = \{z \in M \mid zx = xz \,\forall\, x \in M\}.$$

Definition

A von Neumann algebra M is a *factor* if its center is trivial, i.e., $Z(M) = \mathbb{C}1_M$.

Example:

- $B(\mathcal{H})$ is a factor.
- A commutative von Neumann algebra M is a factor iff $M \cong \mathbb{C}$. In paticular, $L^\infty(\mathbb{R})$ is not a factor.

Factors

The *center* of a von Neumann algebra M is defined by

$$Z(M) = \{z \in M \mid zx = xz \,\forall\, x \in M\}.$$

Definition

A von Neumann algebra M is a *factor* if its center is trivial, i.e., $Z(M) = \mathbb{C}1_M$.

Example:

- $B(\mathcal{H})$ is a factor.
- A commutative von Neumann algebra M is a factor iff $M \cong \mathbb{C}$. In paticular, $L^\infty(\mathbb{R})$ is not a factor.

Informally, factors are building blocks of von Neumann algebras.

Factors

The *center* of a von Neumann algebra M is defined by

$$Z(M) = \{z \in M \mid zx = xz \, \forall \, x \in M\}.$$

Definition

A von Neumann algebra M is a *factor* if its center is trivial, i.e., $Z(M) = \mathbb{C}1_M$.

Example:

- $B(\mathcal{H})$ is a factor.
- A commutative von Neumann algebra M is a factor iff $M \cong \mathbb{C}$. In paticular, $L^\infty(\mathbb{R})$ is not a factor.

Informally, factors are building blocks of von Neumann algebras.

Theorem (von Neumann, 1949)

Every von Neumann algebra on a separable Hilbert space is a direct integral of factors.

Let G be a countable group.

$$\ell^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \sum_{g \in G} |f(g)|^2 < \infty \right\}.$$

Let G be a countable group.

$$\ell^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \sum_{g \in G} |f(g)|^2 < \infty \right\}.$$

The action of G on $\ell^2(G)$ defined by

$$(g.f)(x) = f(g^{-1}x) \quad \forall\, g, x \in G \quad \forall f \in \ell^2(G)$$

induces the *left regular representation* $G \to U(\ell^2(G))$, which extends to a map $\lambda\colon \mathbb{C}G \to B(\ell^2(G))$ by linearity.

Let G be a countable group.

$$\ell^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \sum_{g \in G} |f(g)|^2 < \infty \right\}.$$

The action of G on $\ell^2(G)$ defined by

$$(g.f)(x) = f(g^{-1}x) \quad \forall g, x \in G \quad \forall f \in \ell^2(G)$$

induces the *left regular representation* $G \to U(\ell^2(G))$, which extends to a map $\lambda\colon \mathbb{C}G \to B(\ell^2(G))$ by linearity.

Definition (Murray–von Neumann, 1943)

The *von Neumann algebra of G*, denoted by $L(G)$, is the closure of the image $\lambda(\mathbb{C}G)$ in $B(\ell^2(G))$ in the topology of pointwise convergence. Equivalently, we can define $L(G) = \lambda(\mathbb{C}G)''$.

Let G be a countable group.

$$\ell^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \sum_{g \in G} |f(g)|^2 < \infty \right\}.$$

The action of G on $\ell^2(G)$ defined by

$$(g.f)(x) = f(g^{-1}x) \quad \forall g, x \in G \quad \forall f \in \ell^2(G)$$

induces the *left regular representation* $G \to U(\ell^2(G))$, which extends to a map $\lambda\colon \mathbb{C}G \to B(\ell^2(G))$ by linearity.

Definition (Murray–von Neumann, 1943)

The *von Neumann algebra of G*, denoted by $L(G)$, is the closure of the image $\lambda(\mathbb{C}G)$ in $B(\ell^2(G))$ in the topology of pointwise convergence. Equivalently, we can define $L(G) = \lambda(\mathbb{C}G)''$.

- If $|G| < \infty$, then $L(G) = \mathbb{C}G$.

Let G be a countable group.

$$\ell^2(G) = \left\{ f\colon G \to \mathbb{C} \;\middle|\; \sum_{g \in G} |f(g)|^2 < \infty \right\}.$$

The action of G on $\ell^2(G)$ defined by

$$(g.f)(x) = f(g^{-1}x) \quad \forall g, x \in G \quad \forall f \in \ell^2(G)$$

induces the *left regular representation* $G \to U(\ell^2(G))$, which extends to a map $\lambda\colon \mathbb{C}G \to B(\ell^2(G))$ by linearity.

Definition (Murray–von Neumann, 1943)

The *von Neumann algebra of G*, denoted by $L(G)$, is the closure of the image $\lambda(\mathbb{C}G)$ in $B(\ell^2(G))$ in the topology of pointwise convergence. Equivalently, we can define $L(G) = \lambda(\mathbb{C}G)''$.

- If $|G| < \infty$, then $L(G) = \mathbb{C}G$.
- If G is infinite, $L(G)$ is difficult to understand. E.g., it is not known whether $L(F_m) \cong L(F_n)$ for $m \neq n$.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

Examples.

1. The trivial group.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

Examples.

1. The trivial group.
2. Non-cyclic free groups.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

Examples.

1. The trivial group.
2. Non-cyclic free groups.
3. Non-abelian free solvable groups, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

Examples.

1. The trivial group.
2. Non-cyclic free groups.
3. Non-abelian free solvable groups, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$.

Non-examples.

1. Non-trivial finite groups.

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

Examples.

1. The trivial group.
2. Non-cyclic free groups.
3. Non-abelian free solvable groups, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$.

Non-examples.

1. Non-trivial finite groups.
2. Groups with non-trivial center.

ICC groups

A group G is said to have *infinite conjugacy classes* (abbreviated *ICC*) if the conjugacy class of every non-trivial element of G is infinite.

Examples.

1. The trivial group.
2. Non-cyclic free groups.
3. Non-abelian free solvable groups, $\mathbb{Z} \operatorname{wr} \mathbb{Z}$.

Non-examples.

1. Non-trivial finite groups.
2. Groups with non-trivial center.

Theorem (Murray–von Neumann)

The von-Neumann algebra of a countable group G is a factor iff G is ICC.

Automorphisms of *-algebras

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

Automorphisms of the form $x \mapsto uxu^{-1}$, where u is a unit, are called *inner* and

$$Out(M) = Aut(M)/Inn(M).$$

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

Automorphisms of the form $x \mapsto uxu^{-1}$, where u is a unit, are called *inner* and

$$Out(M) = Aut(M)/Inn(M).$$

Examples (Automorphisms of group rings).

1. Every automorphism of G extends to an automorphism of $\mathbb{C}G$ by linearity.

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

Automorphisms of the form $x \mapsto uxu^{-1}$, where u is a unit, are called *inner* and

$$Out(M) = Aut(M)/Inn(M).$$

Examples (Automorphisms of group rings).

1. Every automorphism of G extends to an automorphism of $\mathbb{C}G$ by linearity.
2. Let
$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \qquad Char(G) = Hom(G, \mathbb{S}^1).$$

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

Automorphisms of the form $x \mapsto uxu^{-1}$, where u is a unit, are called *inner* and

$$Out(M) = Aut(M)/Inn(M).$$

Examples (Automorphisms of group rings).

1. Every automorphism of G extends to an automorphism of $\mathbb{C}G$ by linearity.
2. Let
$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \qquad Char(G) = Hom(G, \mathbb{S}^1).$$
For every $\rho \in Char(G)$, the map $g \mapsto \rho(g)g$ ($\forall\, g \in G$) extends to a *-preserving automorphism of $\mathbb{C}G$.

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

Automorphisms of the form $x \mapsto uxu^{-1}$, where u is a unit, are called *inner* and

$$Out(M) = Aut(M)/Inn(M).$$

Examples (Automorphisms of group rings).

1. Every automorphism of G extends to an automorphism of $\mathbb{C}G$ by linearity.
2. Let
$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \qquad Char(G) = Hom(G, \mathbb{S}^1).$$
For every $\rho \in Char(G)$, the map $g \mapsto \rho(g)g$ $(\forall\, g \in G)$ extends to a *-preserving automorphism of $\mathbb{C}G$.

This yields an embedding

$$Char(G) \rtimes Aut(G) \hookrightarrow Aut(\mathbb{C}G)$$

Automorphisms of *-algebras

An *automorphism* α of a $*$-algebra M is an automorphism of the underlying algebra such that

$$\alpha(x^*) = \alpha(x)^*.$$

Automorphisms of the form $x \mapsto uxu^{-1}$, where u is a unit, are called *inner* and

$$Out(M) = Aut(M)/Inn(M).$$

Examples (Automorphisms of group rings).

1. Every automorphism of G extends to an automorphism of $\mathbb{C}G$ by linearity.
2. Let
$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}, \qquad Char(G) = Hom(G, \mathbb{S}^1).$$
For every $\rho \in Char(G)$, the map $g \mapsto \rho(g)g$ ($\forall\, g \in G$) extends to a *-preserving automorphism of $\mathbb{C}G$.

This yields an embedding

$$Char(G) \rtimes Aut(G) \hookrightarrow Aut(\mathbb{C}G)$$

and a homomorphism

$$Char(G) \rtimes Out(G) \to Out(\mathbb{C}G).$$

Outer automorphisms of group algebras

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Disproved for fields of characteristic $p > 0$ by Gordam and Murray (2021).

Outer automorphisms of group algebras

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Disproved for fields of characteristic $p > 0$ by Gordam and Murray (2021).

Theorem (Folklore)

Suppose that G is a torsion-free group satisfying the Kaplansky unit conjecture. Then $Out(\mathbb{C}G) \cong Char(G) \rtimes Out(G)$.

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Disproved for fields of characteristic $p > 0$ by Gordam and Murray (2021).

Theorem (Folklore)

Suppose that G is a torsion-free group satisfying the Kaplansky unit conjecture. Then $Out(\mathbb{C}G) \cong Char(G) \rtimes Out(G)$.

Proof. Let $\alpha \in Aut(\mathbb{C}G)$. For any $g \in G$, $\alpha(g)$ is a unit. Hence,

$$\alpha(g) = \rho(g)\gamma(g), \quad \text{where } \rho(g) \in \mathbb{C}, \ \gamma(g) \in G.$$

Outer automorphisms of group algebras

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Disproved for fields of characteristic $p > 0$ by Gordam and Murray (2021).

Theorem (Folklore)

Suppose that G is a torsion-free group satisfying the Kaplansky unit conjecture. Then $Out(\mathbb{C}G) \cong Char(G) \rtimes Out(G)$.

Proof. Let $\alpha \in Aut(\mathbb{C}G)$. For any $g \in G$, $\alpha(g)$ is a unit. Hence,

$$\alpha(g) = \rho(g)\gamma(g), \quad \text{where} \;\; \rho(g) \in \mathbb{C}, \;\; \gamma(g) \in G.$$

Obviously, $\gamma \in Aut(G)$ and $\rho \in Hom(G, \mathbb{C}^{\times})$.

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Disproved for fields of characteristic $p > 0$ by Gordam and Murray (2021).

Theorem (Folklore)

Suppose that G is a torsion-free group satisfying the Kaplansky unit conjecture. Then $Out(\mathbb{C}G) \cong Char(G) \rtimes Out(G)$.

Proof. Let $\alpha \in Aut(\mathbb{C}G)$. For any $g \in G$, $\alpha(g)$ is a unit. Hence,

$$\alpha(g) = \rho(g)\gamma(g), \quad \text{where} \;\; \rho(g) \in \mathbb{C}, \;\; \gamma(g) \in G.$$

Obviously, $\gamma \in Aut(G)$ and $\rho \in Hom(G, \mathbb{C}^\times)$. Since α preserves *, we have

$$1 = \alpha(g \cdot g^*) = \alpha(g) \cdot \alpha(g)^* = (\rho(g)g) \cdot (\overline{\rho(g)}g^{-1}) = \rho(g)\overline{\rho(g)}.$$

Therefore, $\rho \in Char(G)$. Thus, the homomorphism

$$Char(G) \rtimes Out(G) \to Out(\mathbb{C}G)$$

is surjective.

Conjecture (Kaplansky unit conjecture over $\mathbb{C}$)

If G is torsion-free, all units of $\mathbb{C}G$ are of the form cg, where $c \in \mathbb{C}$, $g \in G$.

Disproved for fields of characteristic $p > 0$ by Gordam and Murray (2021).

Theorem (Folklore)

Suppose that G is a torsion-free group satisfying the Kaplansky unit conjecture. Then $Out(\mathbb{C}G) \cong Char(G) \rtimes Out(G)$.

Proof. Let $\alpha \in Aut(\mathbb{C}G)$. For any $g \in G$, $\alpha(g)$ is a unit. Hence,

$$\alpha(g) = \rho(g)\gamma(g), \quad \text{where} \;\; \rho(g) \in \mathbb{C}, \;\; \gamma(g) \in G.$$

Obviously, $\gamma \in Aut(G)$ and $\rho \in Hom(G, \mathbb{C}^\times)$. Since α preserves *, we have

$$1 = \alpha(g \cdot g^*) = \alpha(g) \cdot \alpha(g)^* = (\rho(g)g) \cdot (\overline{\rho(g)}g^{-1}) = \rho(g)\overline{\rho(g)}.$$

Therefore, $\rho \in Char(G)$. Thus, the homomorphism

$$Char(G) \rtimes Out(G) \to Out(\mathbb{C}G)$$

is surjective. Injectivity follows from the Kaplansky unit conjecture.

Amenable vs Kazhdan groups

Definition (von Neumann, 1929)

A group G is *amenable* if there exists $\mu\colon 2^G \to [0,1]$ such that $\mu(G) = 1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and $\mu(gA) = \mu(A)$ for all $A, B \subseteq G$ and all $g \in G$.

Definition (von Neumann, 1929)

A group G is *amenable* if there exists $\mu\colon 2^G \to [0,1]$ such that $\mu(G) = 1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and $\mu(gA) = \mu(A)$ for all $A, B \subseteq G$ and all $g \in G$.

Examples: finite and solvable groups, groups of subexponential growth; closed under subgroups, quotients, extensions, and directed unions.

Amenable vs Kazhdan groups

Definition (von Neumann, 1929)

A group G is *amenable* if there exists $\mu \colon 2^G \to [0,1]$ such that $\mu(G) = 1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and $\mu(gA) = \mu(A)$ for all $A, B \subseteq G$ and all $g \in G$.

Examples: finite and solvable groups, groups of subexponential growth; closed under subgroups, quotients, extensions, and directed unions.

Definition (Kazhdan, 1967)

A group G has *property (T)* if every affine isometric action of G on a real Hilbert space has a fixed point.

Amenable vs Kazhdan groups

Definition (von Neumann, 1929)

A group G is *amenable* if there exists $\mu\colon 2^G \to [0,1]$ such that $\mu(G) = 1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and $\mu(gA) = \mu(A)$ for all $A, B \subseteq G$ and all $g \in G$.

Examples: finite and solvable groups, groups of subexponential growth; closed under subgroups, quotients, extensions, and directed unions.

Definition (Kazhdan, 1967)

A group G has *property (T)* if every affine isometric action of G on a real Hilbert space has a fixed point.

Examples: finite groups, lattices in simple Lie groups of $\mathbb{R}$-rank at least 2 (e.g., $SL(n, \mathbb{Z})$ for $n \geq 3$); closed under quotients and extensions.

Definition (von Neumann, 1929)

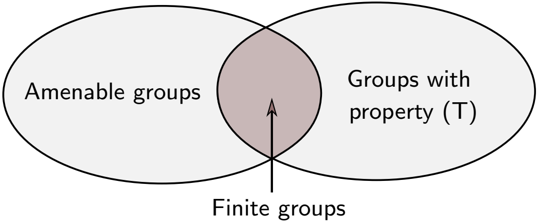
A group G is *amenable* if there exists $\mu\colon 2^G \to [0,1]$ such that $\mu(G)=1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$, and $\mu(gA) = \mu(A)$ for all $A, B \subseteq G$ and all $g \in G$.

Examples: finite and solvable groups, groups of subexponential growth; closed under subgroups, quotients, extensions, and directed unions.

Definition (Kazhdan, 1967)

A group G has *property (T)* if every affine isometric action of G on a real Hilbert space has a fixed point.

Examples: finite groups, lattices in simple Lie groups of $\mathbb{R}$-rank at least 2 (e.g., $SL(n,\mathbb{Z})$ for $n \geq 3$); closed under quotients and extensions.



Alain Connes

Fields Medal (1982) for work on operator algebras

Affiliated with Vanderbilt University in 2005-2014

Theorem (Connes, 1980s)

(a) *Suppose that G is a non-trivial, ICC, amenable group (e.g., $G = \mathbb{Z} \operatorname{wr} \mathbb{Z}$). Then $Out(L(G))$ has cardinality $2^{\aleph_0}$ and contains all countable groups as subgroups.*

(b) *If G is an ICC group with property (T), then $Out(L(G))$ is countable.*

Jones' conjecture

Vaughan Jones

Fields Medal (1990) for work on von Neumann algebras

Worked at Vanderbilt University in 2011-2020

Conjecture (Jones Millennium Problem, 2000)

If G is ICC and has property (T), then $Out(L(G)) \cong Char(G) \ltimes Out(G)$.

Vaughan Jones

Fields Medal (1990) for work on von Neumann algebras

Worked at Vanderbilt University in 2011-2020

Conjecture (Jones Millennium Problem, 2000)

If G is ICC and has property (T), then $Out(L(G)) \cong Char(G) \ltimes Out(G)$.

Since 2000, neither a counterexample nor a single non-trivial example has been found

Vaughan Jones

Fields Medal (1990) for work on von Neumann algebras

Worked at Vanderbilt University in 2011-2020

Conjecture (Jones Millennium Problem, 2000)

If G is ICC and has property (T), then $Out(L(G)) \cong Char(G) \ltimes Out(G)$.

Since 2000, neither a counterexample nor a single non-trivial example has been found and the following problem received considerable attention in recent years.

Problem

Find a non-trivial, ICC group with property (T) satisfying the Jones conjecture.

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow W \stackrel{\varepsilon}{\longrightarrow} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$.

Wreath-like products of groups

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow W \stackrel{\varepsilon}{\longrightarrow} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$. If $I = B$ and the action $B \curvearrowright I$ is by (left) multiplication, we say that W is a *regular wreath-like* product of A and B.

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow W \stackrel{\varepsilon}{\longrightarrow} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$. If $I = B$ and the action $B \curvearrowright I$ is by (left) multiplication, we say that W is a *regular wreath-like* product of A and B.

$$\mathcal{WR}(A, B \curvearrowright I) = \{\text{wreath} - \text{like products of groups } A \text{ and } B \curvearrowright I\}$$

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow W \xrightarrow{\varepsilon} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$. If $I = B$ and the action $B \curvearrowright I$ is by (left) multiplication, we say that W is a *regular wreath-like* product of A and B.

$$\mathcal{WR}(A, B \curvearrowright I) = \{\text{wreath} - \text{like products of groups } A \text{ and } B \curvearrowright I\}$$

$$\mathcal{WR}(A, B) = \{\text{regular wreath} - \text{like products of groups } A \text{ and } B\}$$

Wreath-like products of groups

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow W \stackrel{\varepsilon}{\longrightarrow} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$. If $I = B$ and the action $B \curvearrowright I$ is by (left) multiplication, we say that W is a *regular wreath-like* product of A and B.

$$\mathcal{WR}(A, B \curvearrowright I) = \{\text{wreath} - \text{like products of groups } A \text{ and } B \curvearrowright I\}$$

$$\mathcal{WR}(A, B) = \{\text{regular wreath} - \text{like products of groups } A \text{ and } B\}$$

Examples:

1. $A \times B \in \mathcal{WR}(A, B \curvearrowright \{1\})$.

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i\in I} A_i \longrightarrow W \stackrel{\varepsilon}{\longrightarrow} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$. If $I = B$ and the action $B \curvearrowright I$ is by (left) multiplication, we say that W is a *regular wreath-like* product of A and B.

$$\mathcal{WR}(A, B \curvearrowright I) = \{\mathrm{wreath-like\ products\ of\ groups\ } A \mathrm{\ and\ } B \curvearrowright I\}$$

$$\mathcal{WR}(A, B) = \{\mathrm{regular\ wreath-like\ products\ of\ groups\ } A \mathrm{\ and\ } B\}$$

Examples:

1. $A \times B \in \mathcal{WR}(A, B \curvearrowright \{1\})$.
2. $A \,\mathrm{wr}\, B \in \mathcal{WR}(A, B)$.

Wreath-like products of groups

Definition

A group W is a *wreath-like product* of groups A and B corresponding to an action $B \curvearrowright I$ on an set I if W is an extension of the form

$$1 \longrightarrow \bigoplus_{i \in I} A_i \longrightarrow W \stackrel{\varepsilon}{\longrightarrow} B \longrightarrow 1,$$

where $A_i \cong A$ and

$$wA_iw^{-1} = A_{\varepsilon(w)i}$$

for all $i \in I$ and all $w \in W$. If $I = B$ and the action $B \curvearrowright I$ is by (left) multiplication, we say that W is a *regular wreath-like* product of A and B.

$$\mathcal{WR}(A, B \curvearrowright I) = \{\text{wreath} - \text{like products of groups } A \text{ and } B \curvearrowright I\}$$

$$\mathcal{WR}(A, B) = \{\text{regular wreath} - \text{like products of groups } A \text{ and } B\}$$

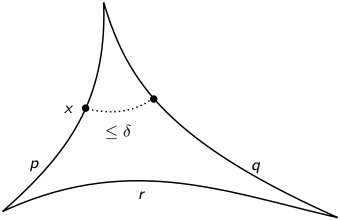
Examples:

1. $A \times B \in \mathcal{WR}(A, B \curvearrowright \{1\})$.
2. $A \,\mathrm{wr}\, B \in \mathcal{WR}(A, B)$.
3. The free metabelian group of rank 2 belongs to $\mathcal{WR}(\mathbb{Z}, \mathbb{Z}^2)$.

Hyperbolic spaces

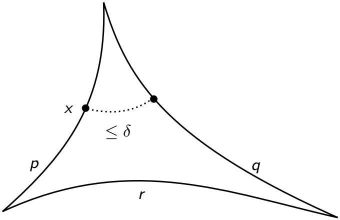
Definition (Gromov)

A geodesic metric space is *hyperbolic* if $\exists\, \delta \geq 0$ such that for any triangle with geodesic sides p, q, r and any $x \in p$, we have $\mathrm{d(x, q \cup r)} \leq \delta$.



Definition (Gromov)

A geodesic metric space is *hyperbolic* if $\exists\,\delta \geq 0$ such that for any triangle with geodesic sides p, q, r and any $x \in p$, we have $\mathrm{d}(x, q \cup r) \leq \delta$.

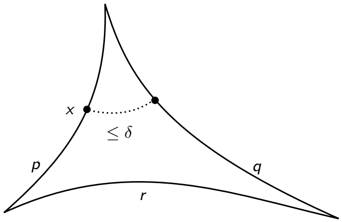


Examples.

1. Any bounded space S is hyperbolic with $\delta = diam(S)$.

Definition (Gromov)

A geodesic metric space is *hyperbolic* if $\exists\, \delta \geq 0$ such that for any triangle with geodesic sides p, q, r and any $x \in p$, we have $\mathrm{d}(x, q \cup r) \leq \delta$.

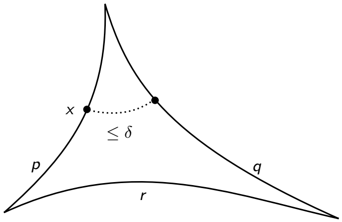


Examples.

1. Any bounded space S is hyperbolic with $\delta = diam(S)$.
2. Any tree is hyperbolic with $\delta = 0$.

Definition (Gromov)

A geodesic metric space is *hyperbolic* if $\exists\,\delta \geq 0$ such that for any triangle with geodesic sides p, q, r and any $x \in p$, we have $\mathrm{d(x, q \cup r)} \leq \delta$.



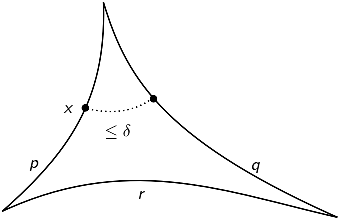
Examples.

1. Any bounded space S is hyperbolic with $\delta = diam(S)$.
2. Any tree is hyperbolic with $\delta = 0$.
3. $\mathbb{H}^n$ is hyperbolic.

Hyperbolic spaces

Definition (Gromov)

A geodesic metric space is *hyperbolic* if $\exists\,\delta \geq 0$ such that for any triangle with geodesic sides p, q, r and any $x \in p$, we have $\mathrm{d}(x, q \cup r) \leq \delta$.



Examples.

1. Any bounded space S is hyperbolic with $\delta = diam(S)$.
2. Any tree is hyperbolic with $\delta = 0$.
3. $\mathbb{H}^n$ is hyperbolic.
4. $\mathbb{R}^n$ is not hyperbolic for $n \geq 2$.

Hyperbolic groups

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Examples.

1. Finite groups.

Hyperbolic groups

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Examples.

1. Finite groups.
2. Free groups of finite rank.

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Examples.

1. Finite groups.
2. Free groups of finite rank.
3. $\pi_1(M)$ for any closed hyperbolic manifold M.

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Examples.

1. Finite groups.
2. Free groups of finite rank.
3. $\pi_1(M)$ for any closed hyperbolic manifold M.
4. $\mathbb{Z}^n$ is not hyperbolic for $n \geq 2$.

Hyperbolic groups

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Examples.

1. Finite groups.
2. Free groups of finite rank.
3. $\pi_1(M)$ for any closed hyperbolic manifold M.
4. $\mathbb{Z}^n$ is not hyperbolic for $n \geq 2$.

Theorem (Chifan–Ioana–Osin–Sun, 2021)

Let $G \in \mathcal{WR}(A, B \curvearrowright I)$, where A is non-trivial abelian, B is hyperbolic, and the action $B \curvearrowright I$ has infinite orbits. If G has property (T), then

$$Out(L(G)) \cong Char(G) \rtimes Out(G).$$

Definition (Gromov, 1980s)

A finitely generated group G is *hyperbolic* if $Cay(G, X)$ is hyperbolic for some finite generating set X of G.

Examples.

1. Finite groups.
2. Free groups of finite rank.
3. $\pi_1(M)$ for any closed hyperbolic manifold M.
4. $\mathbb{Z}^n$ is not hyperbolic for $n \geq 2$.

Theorem (Chifan–Ioana–Osin–Sun, 2021)

Let $G \in \mathcal{WR}(A, B \curvearrowright I)$, where A is non-trivial abelian, B is hyperbolic, and the action $B \curvearrowright I$ has infinite orbits. If G has property (T), then

$$Out(L(G)) \cong Char(G) \rtimes Out(G).$$

Note: If $A \neq 1$ and $A \operatorname{wr} B$ has property (T), then $|B| < \infty$.

Concrete examples of groups satisfying Jones' conjecture

For $S \subseteq G$, let $\langle\langle S \rangle\rangle$ denote the normal closure of S in G.

Concrete examples of groups satisfying Jones' conjecture

For $S \subseteq G$, let $\langle\langle S \rangle\rangle$ denote the normal closure of S in G.

Wreath-like products as quotients:

1. If A is abelian, then $A * B/[\langle\langle A \rangle\rangle, \langle\langle A \rangle\rangle] \cong A \,\mathrm{wr}\, B$.

Concrete examples of groups satisfying Jones' conjecture

For $S \subseteq G$, let $\langle\!\langle S \rangle\!\rangle$ denote the normal closure of S in G.

Wreath-like products as quotients:

1. If A is abelian, then $A * B / [\langle\!\langle A \rangle\!\rangle, \langle\!\langle A \rangle\!\rangle] \cong A \,\mathrm{wr}\, B$.
2. (Cohen–Lyndon, 1963) Let G be a free group, $\langle g \rangle$ a maximal cyclic subgroup of G. For every $n \in \mathbb{N}$, we have

$$G/[\langle\!\langle g^n \rangle\!\rangle, \langle\!\langle g^n \rangle\!\rangle] \in \mathcal{WR}(\mathbb{Z}, G/\langle\!\langle g^n \rangle\!\rangle \curvearrowright I),$$

where $|I| = \infty$ whenever G is non-cyclic.

Concrete examples of groups satisfying Jones' conjecture

For $S \subseteq G$, let $\langle\langle S \rangle\rangle$ denote the normal closure of S in G.

Wreath-like products as quotients:

1. If A is abelian, then $A * B/[\langle\langle A \rangle\rangle, \langle\langle A \rangle\rangle] \cong A \,\mathrm{wr}\, B$.
2. (Cohen–Lyndon, 1963) Let G be a free group, $\langle g \rangle$ a maximal cyclic subgroup of G. For every $n \in \mathbb{N}$, we have

$$G/[\langle\langle g^n \rangle\rangle, \langle\langle g^n \rangle\rangle] \in \mathcal{WR}(\mathbb{Z}, G/\langle\langle g^n \rangle\rangle \curvearrowright I),$$

   where $|I| = \infty$ whenever G is non-cyclic.
3. (Chifan–Ioana–Osin–Sun, 2021) If G is a torsion-free hyperbolic group, the same result holds for all sufficiently large n.

Concrete examples of groups satisfying Jones' conjecture

For $S \subseteq G$, let $\langle\langle S \rangle\rangle$ denote the normal closure of S in G.

Wreath-like products as quotients:

1. If A is abelian, then $A * B / [\langle\langle A \rangle\rangle, \langle\langle A \rangle\rangle] \cong A \,\mathrm{wr}\, B$.
2. (Cohen–Lyndon, 1963) Let G be a free group, $\langle g \rangle$ a maximal cyclic subgroup of G. For every $n \in \mathbb{N}$, we have

$$G/[\langle\langle g^n \rangle\rangle, \langle\langle g^n \rangle\rangle] \in \mathcal{WR}(\mathbb{Z}, G/\langle\langle g^n \rangle\rangle \curvearrowright I),$$

   where $|I| = \infty$ whenever G is non-cyclic.
3. (Chifan–Ioana–Osin–Sun, 2021) If G is a torsion-free hyperbolic group, the same result holds for all sufficiently large n.

Corollary (Chifan–Ioana–Osin–Sun, 2021)

Let G be a torsion-free hyperbolic group with property (T). For any $g \in G \setminus \{1\}$ and any sufficiently large $n \in \mathbb{N}$, the group

$$G/[\langle\langle g^n \rangle\rangle, \langle\langle g^n \rangle\rangle]$$

is non-trivial, ICC, has property (T), and satisfies the Jones conjecture.

Realization problem for outer automorphisms

Recall the following.

Theorem (Connes, 1980)

For any ICC group G with property (T), $Out(L(G))$ is countable.

Realization problem for outer automorphisms

Recall the following.

Theorem (Connes, 1980)

For any ICC group G with property (T), $\mathrm{Out}(L(G))$ is countable.

It is natural to ask the following.

Problem

Which countable groups can be realized as $\mathrm{Out}(L(G))$ for an ICC group G with property (T)?

Realization problem for outer automorphisms

Recall the following.

Theorem (Connes, 1980)

For any ICC group G with property (T), $Out(L(G))$ is countable.

It is natural to ask the following.

Problem

Which countable groups can be realized as $Out(L(G))$ for an ICC group G with property (T)?

No results since 1980: not a single example was known of a countable group that realizes as $Out(L(G))$ for a non-trivial, ICC group G with property (T).

Recall the following.

Theorem (Connes, 1980)

For any ICC group G with property (T), $Out(L(G))$ is countable.

It is natural to ask the following.

Problem

Which countable groups can be realized as $Out(L(G))$ for an ICC group G with property (T)?

No results since 1980: not a single example was known of a countable group that realizes as $Out(L(G))$ for a non-trivial, ICC group G with property (T).

Theorem (Chifan–Ioana–Osin–Sun, 2023)

For any countable group Q, there exists a non-trivial, ICC group G with property (T) such that $Out(L(G)) \cong Q$.

References

1. I. Chifan, A. Ioana, D. Osin, B. Sun, Wreath-like products of groups and their von Neumann algebras I: W^*-superrigidity, *arXiv:2111.04708*; to appear in *Annals of Math.*
2. I. Chifan, A. Ioana, D. Osin, B. Sun, Wreath-like products of groups and their von Neumann algebras II: Outer automorphisms, *arXiv:2304.07457.*

References

1. I. Chifan, A. Ioana, D. Osin, B. Sun, Wreath-like products of groups and their von Neumann algebras I: W^*-superrigidity, *arXiv:2111.04708*; to appear in *Annals of Math.*

2. I. Chifan, A. Ioana, D. Osin, B. Sun, Wreath-like products of groups and their von Neumann algebras II: Outer automorphisms, *arXiv:2304.07457.*

Thank you!