Generation of the second maximal subgroups of the symmetric groups

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June 8, 2023

Introduction

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Let T be a finite, simple group. Then

$$d(T) \leq 2.$$

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Theorem (Dalla Volta, Lucchini, 1994)

If R is an almost simple group, then

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Theorem (Lucchini, Marion, Tracey, 2019)

If R is an almost simple group, and $M <_{\max} R$ then

$$d(M) \leq 5,$$

which is sharp.

Second maximal subgroups

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All such K have

$$\mathrm{soc}(R) = L_2(q), {}^2B_2(q), {}^2G_2(q)$$

and M a Borel subgroup.

Second maximal subgroups

Theorem (Burness, Liebeck, Shalev, 2016)

Let $K <_{\max} M <_{\max} R$ where R is an almost simple group. Then either

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2. $d(K) \leq 70$.

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$$d(G) \leq d(G/N) + d(N).$$

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For example, by induction:

$$d(A_k^t) \leq 2t \qquad \forall k \geq 5.$$

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However, in return it provides much better bounds. For example:

$$d(A_k^t) \leq \begin{cases} 2 & \text{if } t \leq 17 \\ 3 & \text{if } t \leq 1060 \\ \vdots & \end{cases} \qquad \forall k \geq 5.$$

Second maximal subgroups of S_n

Theorem (Burness, Liebeck, Shalev, 2016)

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Theorem (Burness, Liebeck, Shalev, 2016)

Let K be a second maximal subgroup of R, with R almost simple and $soc(R) = A_n$. Then

$$d(K) \leq 10.$$

Second maximal subgroups of S_n

For simplicity, let $R = S_n$.

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Theorem (O'Nan-Scott)

The maximal subgroups of R fall into the following families:

(I) *intransitive,* $S_k \times S_{n-k}$,

(II) *affine,* $AGL_d(p)$,

(III) *wreath product,* $S_k \wr S_t$,

(IV) *diagonal type,* $T^k.(Out(T) \times S_k)$,

(V) *almost simple.*

The maximal subgroups of $S_k \times S_{n-k}$ are

- $J \times S_{n-k}$ for $J <_{\max} S_k$,
- $S_k \times J$ for $J <_{\max} S_{n-k}$, or
- $(A_k \times A_{n-k}).2$.

Classifying second maximal subgroups

Family (II): Affine

The maximal subgroups of $\mathrm{AGL}_d(p) = \mathbb{F}_p^d \rtimes \mathrm{GL}(\mathbb{F}_p^d)$ are

- $\mathbb{F}_p^d \rtimes J$ for $J <_{\max} \mathrm{GL}(\mathbb{F}_p^d)$, or
- Isomorphic to $\mathrm{GL}(\mathbb{F}_p^d)$.

The maximal subgroups of $S_k \wr S_t$ are

- $S_k \wr J$ for $J <_{\max} S_t$
- $J \wr S_t$ for $J <_{\max} S_k$, $J \neq A_k$, or
- in one of three other families, all of which contain A_k^t.

The maximal subgroups of $T^k.(\mathrm{Out}(T) \times S_k)$ are

- $T^k.J$ for $J <_{\max} \mathrm{Out}(T) \times S_k$, or
- $(J \cap T)^k.(\mathrm{Out}(T) \times S_k)$ for $J <_{\max} \mathrm{Aut}(T)$.

Classifying second maximal subgroups

Family (V): Almost simple

The maximal subgroups of the almost simple groups have been treated by Lucchini, Marion and Tracey, so

$$d(K) \leq 5$$

in this case.

Results

Theorem (M.C.)

Let K be a second maximal subgroup of S_n or A_n. Then

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Let $T := \mathrm{PSL}_4(9)$. Let $M := T^k.(\mathrm{Out}(T) \times S_k) <_{\max} S_n$, with $k := |T|$.

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Then

$$T^k.(\mathrm{Out}(T) \times J) <_{\max} M <_{\max} S_n,$$

and

$$d(T^k.(\mathrm{Out}(T) \times J)) = 7.$$

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for T one of

$$\mathrm{PSL}_n(q), P\Omega^{\pm}_{2m}(q), E_6(q).$$

Thanks for listening!