Group rings with (centrally) metabelian unit groups

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More than 30 years ago

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A group G is metabelian, if it has an abelian normal subgroup A such that G/A is also abelian, or equivalently,

- G' is abelian, or
- G'' is trivial, or
- G is solvable of derived length at most 2.

Obviously, every abelian group is metabelian, S_3 is metabelian but not abelian, S_4 is not metabelian.

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If F is a field of characteristic $p > 2$, and G is a finite group, then $U(FG)$ is metabelian if, and only if,

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"The delicate case $p = 2$ seems to require a separate discussion." – This is attested by J. Kurdics (1996, *Period. Math. Hung.* **32**) and independently, by D.B. Coleman and R. Sandling (1998, *J. Pure Appl. Algebra* **131**)

In the sequel, by F we always mean a field of characteristic $p > 2$. The aim of this presentation is to remove the restriction that G is finite.

Group algebras with solvable unit groups

Motose and Tominaga (1968), Bateman (1971), Motose and Ninomiya (1972), A. Bovdi and Khripta (1974), Taylor (1975), A. Bovdi and Khripta (1977), Passman (1977), A. Bovdi (1992)

The final result was obtained by A. Bovdi (2005, *Commun. Algebra* **33**). (Of course, the usual restriction that G modulo its torsion part be a u.p. group must be imposed for the sufficiency.)

When G is torsion, $U(FG)$ is solvable if, and only if, G is a finite p-group, provided $|F| > 3$. The $|F| = 3$ case and the characterization for non-torsion groups are more involved.

Adalbert Bovdi (Béla Bódi) 1935 – 2023

The derived length of the group of units

On the assumption that $U(FG)$ is solvable, it is natural to ask about its derived length, $\mathrm{dl}(U(FG))$, but the picture is not as clear here. It seems quite difficult to give a general formula, and just a few results have been proved.

Theorem (C. Baginski (2002))

Let G be a finite p-group. If G' is cyclic, then $\mathrm{dl}(U(FG)) = \lceil \log_2(|G'| + 1) \rceil$.

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Z. Balogh and Y. Li (2007):

1. If G is torsion and nilpotent, then $\mathrm{dl}(U(FG))$ is still equal to $\lceil \log_2(|G'| + 1) \rceil$.
2. For non-nilpotent G, a more involved formula is given. In this case $\mathrm{dl}(U(FG))$ is equal to either $\lceil \log_2(|G'| + 1) \rceil$ or $\lceil \log_2(|G'| + 1) \rceil + 1$.

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What happens if G is nilpotent, but non-torsion?

The non-torsion case is more involved

Theorem (J. (2017))

Let G be a non-abelian nilpotent group such that G' is a finite abelian p-group.

1. *If* $G' = \mathrm{Syl}_p(G)$ *and* $\gamma_3(G) \subseteq (G')^p$, *then* $\mathrm{dl}(U(FG)) \leq \lceil \log_2(\frac{2}{3}(t(G')+1)) \rceil$;
2. *If* G' *is cyclic, then* $\mathrm{dl}(U(FG)) \geq \lceil \log_2(\frac{2}{3}(t(G')+1)) \rceil$.

In particular, if $G' = \mathrm{Syl}_p(G)$ *is cyclic, then* $\mathrm{dl}(U(FG)) = \lceil \log_2(\frac{2}{3}(|G'|+1)) \rceil$.

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2. *If* G' *is cyclic, then* $\mathrm{dl}(U(FG)) \geq \lceil \log_2(\frac{2}{3}(t(G')+1)) \rceil$*.*

In particular, if $G' = \mathrm{Syl}_p(G)$ *is cyclic, then* $\mathrm{dl}(U(FG)) = \lceil \log_2(\frac{2}{3}(|G'|+1)) \rceil$*.*

Consequently, if G is nilpotent of class 2 and

- $p = 5$ and $G' = \mathrm{Syl}_p(G)$, then $\mathrm{dl}(U(FG)) = 2$, and if
- $p = 3$ and $G' = \mathrm{Syl}_p(G)$ is elementary abelian of order p^2, then $\mathrm{dl}(U(FG)) = 2$.

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If G is non-abelian nilpotent, then the condition $G' = \mathrm{Syl}_p(G)$ is satisfied only if G is non-torsion.

The lower bound on the derived length of the unit group

Theorem (C. Baginski (2002))

Let G be a finite p-group. If G' is cyclic, then $\mathrm{dl}(U(FG)) = \lceil \log_2(|G'| + 1) \rceil$.

Corollary (C. Baginski (2002))

Let G be a finite non-abelian p-group. Then $\mathrm{dl}(U(FG)) \geq \lceil \log_2(p + 1) \rceil$.

Group algebras with solvable unit groups of minimal derived length I

Theorem (F. Catino and E. Spinelli (2010))

Let G be a non-abelian torsion nilpotent group. Then $\mathrm{dl}(U(FG)) \geq \lceil \log_2(p+1) \rceil$, with equality, if and only if, G' has order p.

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Let G be a torsion group. Then $U(FG)$ is metabelian if, and only if, either G is abelian, or $p = 3$ and G' is central of order 3.

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What happens if G is non-torsion?

Group algebras with solvable unit groups of minimal derived length II

If G is non-torsion and G' is a p-group, then $\lceil \log_2(\frac{2}{3}(p+1)) \rceil$ is the lower bound.

Theorem (J., G.T. Lee, S.K. Sehgal and E. Spinelli (2020))

Let G be a non-abelian nilpotent group such that FG is modular. If G has an element of infinite order, then

1. $\mathrm{dl}(U(FG)) > \lceil \log_2(p+1) \rceil$ *if G' is not a finite p-group, and*
2. $\mathrm{dl}(U(FG)) \geq \lceil \log_2(\frac{2}{3}(p+1)) \rceil$ *otherwise. Furthermore,* $\mathrm{dl}(U(FG)) > \lceil \log_2(\frac{2}{3}(p+1)) \rceil$ *if $p > 3$ and $|G'| = p^n$ for some $n > 1$.*

Main result I

Theorem (J. and E. Spinelli (2021))

Let G be a non-abelian group such that FG is modular. Then $U(FG)$ is metabelian if, and only if, G is nilpotent of class 2 and either

1. $p = 3$ *and* G' *has order* p,
2. $p = 3$ *and* $G' = \mathrm{Syl}_p(G)$ *is elementary abelian of order* p^2, *or*
3. $p = 5$ *and* $G' = \mathrm{Syl}_p(G)$ *has order* p.

Cenrally metabelian groups

A group G is said to be centrally metabelian (or, alternately, centre-by-metabelian), if G has a normal subgroup H, which is central in G and the factor group G/H is metabelian. or, equivalently, G'' is (no longer necessarily trivial, but) central in G.

Consequently, every metabelian group is centrally metabelian, and if G is centrally metabelian, then $\mathrm{dl}(G) \leq 3$.

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Theorem (M. Sahai (1996, *Publ. Mat., Barc.* 40))

If G is a finite group, then $U(FG)$ is centrally metabelian if, and only if, either G is abelian, or $p = 3$ and G' has order 3.

Main result II

Theorem (J. and M. Sahai (2023))

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Thank you!