σ-Subnormality in Locally Finite Groups

Maria Ferrara

Advances in Group Theory and Applications 2023

June 5–9, 2023

AGTA

Università
degli Studi
della Campania
Luigi Vanvitelli

References

- M.F., Marco Trombetti
  σ-Subnormality in locally finite groups
  Journal of Algebra, Vol. 614 (2023), 867-897

- M.F., Marco Trombetti
  Joins of σ-Subnormal Subgroups
  Submitted

First results on joins

Helmut Wielandt
(1910 - 2001)

First results on joins

- It is easy to see that the intersection of any finite set of subnormal subgroups is itself subnormal.

First results on joins

- It is easy to see that the intersection of any finite set of subnormal subgroups is itself subnormal.

- *Wielandt tells us that Robert Remak asked, in a seminar in the mid-1930s, whether, in a finite group, the subgroup generated by two subnormal subgroups (that is their join) is itself always subnormal.*

First results on joins

- It is easy to see that the intersection of any finite set of subnormal subgroups is itself subnormal.

- *Wielandt tells us that Robert Remak asked, in a seminar in the mid-1930s, whether, in a finite group, the subgroup generated by two subnormal subgroups (that is their join) is itself always subnormal.*

Subnormal Subgroups of Groups
John C. Lennox &
Stewart E. Stonehewer
Oxford Univ. Press (1987)

First results on joins

- *Wielandt tells us that Robert Remak asked, in a seminar in the mid-1930s, whether, in a finite group, the subgroup generated by two subnormal subgroups (that is their join) is itself always subnormal.*

YES!
H. Wielandt
Eine Verallgemeinerung der invarianten Untergruppen
Math. Z., **45**, 209–244 (1939)

First results on joins

- Zassenhaus constructed an example in 1958 which shows that a join of two subnormal subgroups can fail to be subnormal in an infinite group!

First results on joins

- Zassenhaus constructed an example in 1958 which shows that a join of two subnormal subgroups can fail to be subnormal in an infinite group!
- Zassenhaus wrote this example (as an exercise) on page 235 of the book

The theory of groups, 2nd ed.
New York: Chelsea 1958

First results on joins

- However, in a footnote of his paper, Wielandt states that his proof could be applied to the case where G is not necessarily finite, provided that G satisfies *Max-sn*.

First results on joins

- However, in a footnote of his paper, Wielandt states that his proof could be applied to the case where G is not necessarily finite, provided that G satisfies *Max-sn*.

- Recall that a group G satisfies *Max-sn* if every non-empty set of subnormal subgroups of G contains at least one maximal member;

First results on joins

- However, in a footnote of his paper, Wielandt states that his proof could be applied to the case where G is not necessarily finite, provided that G satisfies *Max-sn*.
- Recall that a group G satisfies *Max-sn* if every non-empty set of subnormal subgroups of G contains at least one maximal member; or equivalently every strictly ascending chain of subnormal subgroups of G has finite length.

First results on joins

- However, in a footnote of his paper, Wielandt states that his proof could be applied to the case where G is not necessarily finite, provided that G satisfies *Max-sn*.
- Recall that a group G satisfies *Max-sn* if every non-empty set of subnormal subgroups of G contains at least one maximal member; or equivalently every strictly ascending chain of subnormal subgroups of G has finite length.
- This result will be extended to many classes of infinite groups under suitable hypothesis.

First results on joins

- After Wielandt's theorem one of the major steps forward in the join problem was made by Roseblade in 1964.

First results on joins

- After Wielandt's theorem one of the major steps forward in the join problem was made by Roseblade in 1964.
- He showed that if subnormal subgroups H and K satisfy *Min-sn* (the minimal condition for subnormal subgroups) then their join J is also subnormal.

J.E. Roseblade
On certain subnormal coalition classes
J. Algebra. **1**, 132-138 (1964).

First results on joins

- The following year Roseblade obtained the same conclusion when H and K satisfy *Max-sn* (the maximal condition for subnormal subgroups).

J.E. Roseblade
A note on subnormal coalition classes
Math. Z. **90**, 373-375 (1965).

First results on joins

- The following year Roseblade obtained the same conclusion when H and K satisfy *Max-sn* (the maximal condition for subnormal subgroups).

J.E. Roseblade
A note on subnormal coalition classes
Math. Z. **90**, 373-375 (1965).

- However, these and other results established rather more than the subnormality of J by showing how the internal structure of H and K restricts that of J.

The class $\mathfrak{S}$

- Robinson denoted by $\mathfrak{S}$ the class of all groups in which the join of two subnormal subgroups is always subnormal.

The class $\mathfrak{S}$

- Robinson denoted by $\mathfrak{S}$ the class of all groups in which the join of two subnormal subgroups is always subnormal.
- $\mathfrak{S}$ contains all groups G such that
  1. G' is nilpotent,
  2. G' satisfies *Max-sn*.

The subnormal join property (SJP)

- A group is said to have the subnormal join property (SJP) if the join of every pair - and hence of every finite set - of subnormal subgroups is subnormal.

The subnormal join property (SJP)

- A group is said to have the subnormal join property (SJP) if the join of every pair - and hence of every finite set - of subnormal subgroups is subnormal.
- Now the set of all subnormal subgroups of a group is a partially ordered subset of the lattice of all subgroups and is moreover closed under finite intersections.

The subnormal join property (SJP)

- A group is said to have the subnormal join property (SJP) if the join of every pair - and hence of every finite set - of subnormal subgroups is subnormal.
- Now the set of all subnormal subgroups of a group is a partially ordered subset of the lattice of all subgroups and is moreover closed under finite intersections.
- Hence a group has the SJP exactly when the set of all its subnormal subgroups is a sublattice of the lattice of subgroups.

The class $\mathfrak{S}^\infty$

- We denote with $\mathfrak{S}^\infty$ the class of groups in which all joins of arbitrarily many subnormal subgroups are subnormal.

The class $\mathfrak{S}^\infty$

- We denote with $\mathfrak{S}^\infty$ the class of groups in which all joins of arbitrarily many subnormal subgroups are subnormal.
- $\mathit{Max\text{-}sn} \leq \mathfrak{S}^\infty$

The class $\mathfrak{S}^\infty$

- We denote with $\mathfrak{S}^\infty$ the class of groups in which all joins of arbitrarily many subnormal subgroups are subnormal.
- $\textit{Max-sn} \leq \mathfrak{S}^\infty$
- $\textit{Min-sn} \leq \mathfrak{S}^\infty$

The class $\mathfrak{S}^\infty$

- We denote with $\mathfrak{S}^\infty$ the class of groups in which all joins of arbitrarily many subnormal subgroups are subnormal.
- *Max-sn* $\leq \mathfrak{S}^\infty$
- *Min-sn* $\leq \mathfrak{S}^\infty$

- We denote by $\mathfrak{M}$ the class of all groups having a finite series whose factors are in *Min-sn* or in *Max-sn*, i.e. satisfy the minimal or the maximal condition on subnormal subgroups.

The class $\mathfrak{S}^\infty$

- We denote with $\mathfrak{S}^\infty$ the class of groups in which all joins of arbitrarily many subnormal subgroups are subnormal.
- *Max-sn* $\leq \mathfrak{S}^\infty$
- *Min-sn* $\leq \mathfrak{S}^\infty$

- We denote by $\mathfrak{M}$ the class of all groups having a finite series whose factors are in *Min-sn* or in *Max-sn*, i.e. satisfy the minimal or the maximal condition on subnormal subgroups.

- $\mathfrak{M} \leq \mathfrak{S}^\infty$

The class $\mathfrak{S}^\infty$

- Note that soluble groups in the class $\mathfrak{M}$ have finite rank.

The class $\mathfrak{S}^\infty$

- Note that soluble groups in the class $\mathfrak{M}$ have finite rank.
- A group is said to have finite rank r if any of its finitely generated subgroups can be generated by r elements.

The class $\mathfrak{S}^\infty$

- Note that soluble groups in the class $\mathfrak{M}$ have finite rank.
- A group is said to have finite rank r if any of its finitely generated subgroups can be generated by r elements.
- There is an example (due to Smith) of a soluble group of finite rank which is not in the class $\mathfrak{S}^\infty$.

The join problem - a criterion

- If n is an integer, we will denote with $\pi(n)$ the set of all primes diving n.

The join problem - a criterion

- If n is an integer, we will denote with $\pi(n)$ the set of all primes diving n.
- If G is a finite group, $\pi(G) = \pi(|G|)$ will be the set of all primes dividing the order of G.

The join problem - a criterion

- If n is an integer, we will denote with $\pi(n)$ the set of all primes diving n.
- If G is a finite group, $\pi(G) = \pi(|G|)$ will be the set of all primes dividing the order of G.
- In general, $\pi(G)$ denotes the set of primes dividing the orders of elements of finite order in G.

The join problem - a criterion

- If n is an integer, we will denote with $\pi(n)$ the set of all primes diving n.
- If G is a finite group, $\pi(G) = \pi(|G|)$ will be the set of all primes dividing the order of G.
- In general, $\pi(G)$ denotes the set of primes dividing the orders of elements of finite order in G.
- If π is any set of primes, a π-group is just a group in which $\pi(G) \subseteq \pi$.

The join problem - a criterion

- Suppose that H and K are subnormal subgroups of a group G.

The join problem - a criterion

- Suppose that H and K are subnormal subgroups of a group G.
- Wielandt proved that H and K permute provided that G satisfies *Min-sn* and $\pi(H/H') \cap \pi(K/K') = \emptyset$.

The join problem - a criterion

- Suppose that H and K are subnormal subgroups of a group G.
- Wielandt proved that H and K permute provided that G satisfies *Min-sn* and $\pi(H/H') \cap \pi(K/K') = \emptyset$.
- Now, $\pi(H/H') \cap \pi(K/K') = \emptyset$, for groups with *Min-sn*, implies that the tensor product $H/H' \otimes K/K'$ is trivial.

The join problem - a criterion

- Suppose that H and K are subnormal subgroups of a group G.
- Wielandt proved that H and K permute provided that G satisfies *Min-sn* and $\pi(H/H') \cap \pi(K/K') = \emptyset$.
- Now, $\pi(H/H') \cap \pi(K/K') = \emptyset$, for groups with *Min-sn*, implies that the tensor product $H/H' \otimes K/K'$ is trivial.
- When $H/H' \otimes K/K'$ is trivial, we say that the subgroups H and K are orthogonal ($H \perp K$).

The join problem - a criterion

- In the light of Wielandt's result, Philip Hall conjectured that any pair of orthogonal subnormal subgroups of an arbitrary group permute.
- Roseblade established the truth of this conjecture in 1965.
- **If H and K are orthogonal subnormal subgroups of a group, then** $HK = KH$.
- (J.C. Lennox, S.E. Stonehewer): $HK = KH$ is a sufficient condition for $J = \langle H, K \rangle$ to be subnormal in G but it is not necessary (see Theorem 1.2.5 of the book *Subnormal Subgroups of Groups*).

σ-Subnormality

- Let $\sigma = \{\sigma_i : i \in I\}$ be a partition of the set $\mathbb{P}$ of all primes

  1. $\mathbb{P} = \bigcup_{i \in I} \sigma_i$,
  2. $\sigma_i \cap \sigma_j = \emptyset$ for all $i \neq j$,
  3. $\sigma_i \neq \emptyset$ for all $i \in I$.

- G is σ-primary if G is a σ_i-group for some $i \in I$.

σ-subnormality

- A subgroup H of G is σ-subnormal in G if there is a subgroup chain

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

1. H_{i-1} is normal in H_i or

σ-subnormality

- A subgroup H of G is σ-subnormal in G if there is a subgroup chain

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

1. H_{i-1} is normal in H_i or
2. $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$.

σ-subnormality

- A subgroup H of G is σ-subnormal in G if there is a subgroup chain

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

1. H_{i-1} is normal in H_i or
2. $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$.

- $H_i/(H_{i-1})_{H_i}$ is σ-primary if $H_i/(H_{i-1})_{H_i}$ is a σ_i-group for some $i \in I$.

σ-subnormality

A. N. Skiba
On σ-subnormal and σ-permutable subgroups of finite groups
J. Algebra **436**, 1-16 (2015).

- Skiba studied the main properties of σ-subnormal subgroups in finite groups.

σ-subnormality

A. N. Skiba
On σ-subnormal and σ-permutable subgroups of finite groups
J. Algebra **436**, 1-16 (2015).

- Skiba studied the main properties of σ-subnormal subgroups in finite groups.
- In particular, he showed that the set of all σ-subnormal subgroups has a strong influence on the structure of a finite soluble group.

Join of σ-subnormal subgroups

- This led many authors to investigate which of the most relevant theorems about subnormal subgroups have analogs in terms of σ-subnormal subgroups.
- It turns out that, for instance, that the join of σ-subnormal subgroups (of a finite group) is σ-subnormal.

Join of σ-subnormal subgroups

A. Ballester-Bolinches, S.F. Kamornikov
On σ-subnormality criteria in finite groups
J. Pure Apple. Algebra **226** (2), 106822 (2022).

A. Ballester-Bolinches, S.F. Kamornikov, M.C. Pedraza-Aguilera, X. Yi
On σ-subnormal subgroups of factorised finite groups
J. Algebra **559**, 195-202 (2020).

From Finite groups to Infinite groups

From Finite groups to Infinite groups

- Moving from finite to arbitrary infinite groups, the intermediate step is that of **locally finite** groups.

From Finite groups to Infinite groups

- Moving from finite to arbitrary infinite groups, the intermediate step is that of **locally finite** groups.
- These are groups in which every finite subset generates a finite subgroup.

From Finite groups to Infinite groups

- In fact, in such kind of a group G, it makes sense to replace the "order of G" by "$\pi(G)$" that is the set of all primes p such that G has elements of order p.

From Finite groups to Infinite groups

- In fact, in such kind of a group G, it makes sense to replace the "order of G" by "$\pi(G)$" that is the set of all primes p such that G has elements of order p.
- If π is any set of primes, a π-group is just a group in which $\pi(G) \subseteq \pi$.

From Finite groups to Infinite groups

- In fact, in such kind of a group G, it makes sense to replace the "order of G" by "$\pi(G)$" that is the set of all primes p such that G has elements of order p.
- If π is any set of primes, a π-group is just a group in which $\pi(G) \subseteq \pi$.
- A subgroup H of G is σ-subnormal in G if there is a subgroup chain

$$H = H_0 \leq H_1 \leq \ldots \leq H_n = G$$

1. H_{i-1} is normal in H_i or
2. $H_i/(H_{i-1})_{H_i}$ is σ-primary for all $i = 1, \ldots, n$
(remember that a group G is σ-primary if G is a σ_i-group for some $i \in I$).

Locally finite groups

- Although in finite groups, σ-subnormal subgroups form a sublattice of the lattice of all subgroups, this is no longer true for locally finite groups!

Locally finite groups

- Although in finite groups, σ-subnormal subgroups form a sublattice of the lattice of all subgroups, this is no longer true for locally finite groups!
- We provide many criteria to determining when a subgroup is σ-subnormal starting from the much weaker concept of σ-seriality.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K be σ-subnormal of G and put $J = \langle H, K \rangle$.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K be σ-subnormal of G and put $J = \langle H, K \rangle$.
- If the join of any family of subnormal subgroups of G contained in J is subnormal in G, then J is σ-subnormal in G.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K are σ-subnormal subgroups of G such that $J = HK = KH$, then J is σ-subnormal in G.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K are σ-subnormal subgroups of G such that $J = HK = KH$, then J is σ-subnormal in G.
- Note that this holds in particular if H normalizes K.

The class $\mathfrak{S}^\infty$

- We denote with $\mathfrak{S}^\infty$ the class of groups in which all joins of arbitrarily many subnormal subgroups are subnormal.
- *Max-sn* $\leq \mathfrak{S}^\infty$
- *Min-sn* $\leq \mathfrak{S}^\infty$

- We denote by $\mathfrak{M}$ the class of all groups having a finite series whose factors are in *Min-sn* or in *Max-sn*, i.e. satisfy the minimal or the maximal condition on subnormal subgroups.

- $\mathfrak{M} \leq \mathfrak{S}^\infty$

The class $\mathfrak{S}_\sigma^\infty$

- We denote with $\mathfrak{S}_\sigma^\infty$ the class of groups in which all joins of arbitrarily many σ-subnormal subgroups are subnormal.

The class $\mathfrak{S}_\sigma^\infty$

- We denote with $\mathfrak{S}_\sigma^\infty$ the class of groups in which all joins of arbitrarily many σ-subnormal subgroups are subnormal.
- $\textit{Max-sn}_\sigma \leq \mathfrak{S}_\sigma^\infty$

The class $\mathfrak{S}_\sigma^\infty$

- We denote with $\mathfrak{S}_\sigma^\infty$ the class of groups in which all joins of arbitrarily many σ-subnormal subgroups are subnormal.
- $\mathit{Max}\text{-}\mathit{sn}_\sigma \leq \mathfrak{S}_\sigma^\infty$
- $\mathit{Min}\text{-}\mathit{sn}_\sigma \leq \mathfrak{S}_\sigma^\infty$

The class $\mathfrak{S}_\sigma^\infty$

- We denote with $\mathfrak{S}_\sigma^\infty$ the class of groups in which all joins of arbitrarily many σ-subnormal subgroups are subnormal.
- *Max-sn*$_\sigma \leq \mathfrak{S}_\sigma^\infty$
- *Min-sn*$_\sigma \leq \mathfrak{S}_\sigma^\infty$

- We denote by $\mathfrak{M}_\sigma$ the class of all groups having a finite series whose factors are in *Min-sn*$_\sigma$ or in *Max-sn*$_\sigma$, i.e. satisfy the minimal or the maximal condition on σ-subnormal subgroups.

The class $\mathfrak{S}_\sigma^\infty$

- We denote with $\mathfrak{S}_\sigma^\infty$ the class of groups in which all joins of arbitrarily many σ-subnormal subgroups are subnormal.
- $\textit{Max-sn}_\sigma \leq \mathfrak{S}_\sigma^\infty$
- $\textit{Min-sn}_\sigma \leq \mathfrak{S}_\sigma^\infty$
- We denote by $\mathfrak{M}_\sigma$ the class of all groups having a finite series whose factors are in $\textit{Min-sn}_\sigma$ or in $\textit{Max-sn}_\sigma$, i.e. satisfy the minimal or the maximal condition on σ-subnormal subgroups.
- $\mathfrak{M}_\sigma \leq \mathfrak{S}_\sigma^\infty$

Roseblade's Theorem

- If H and K are orthogonal subnormal subgroups of a group, then $HK = KH$.

Roseblade's Theorem

- If H and K are orthogonal subnormal subgroups of a group, then $HK = KH$.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K be orthogonal σ-subnormal subgroups of G.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K be orthogonal σ-subnormal subgroups of G.
- Then $J = \langle H, K \rangle$ is σ-subnormal in G.

M.F., Marco Trombetti

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group.
- Let H and K be orthogonal σ-subnormal subgroups of G.
- Then $J = \langle H, K \rangle$ is σ-subnormal in G.

- Although Roseblade's result shows that orthogonal subnormal subgroups permute, this is no longer true for orthogonal σ-subnormal subgroups.

M.F., Marco Trombetti

- A group is said to have finite rank r if any of its finitely generated subgroups can be generated by r elements.

M.F., Marco Trombetti

- A group is said to have finite rank r if any of its finitely generated subgroups can be generated by r elements.
- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.

M.F., Marco Trombetti

- A group is said to have finite rank r if any of its finitely generated subgroups can be generated by r elements.

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group of finite rank.

M.F., Marco Trombetti

- A group is said to have finite rank r if any of its finitely generated subgroups can be generated by r elements.

- Let $\sigma = \{\sigma_i \ : \ i \in I\}$ be a partition of $\mathbb{P}$.
- Let G be a locally finite group of finite rank.
- Then $G \in \mathfrak{S}_\sigma$.