From simple and exceptional Lie algebras towards solvable and nilpotent ones

Cristina Draper Fontanals

UNIVERSIDAD DE MÁLAGA

Joint work with Juana Sánchez and Thomas Meyer

June 5–9, 2023, Lecce (Italy); Advances in Group Theory and Applications 2023

Lie algebras (Setting)

Definition. A Lie algebra $(\mathfrak{g}, [\ ,\ ])$ over a field (today $\mathbb{F} = \mathbb{R}, \mathbb{C}$) is an algebra satisfiying

♡ Skewsymmetry: $[x, y] = -[y, x]$;

♡ Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Definition. A Lie group G is a group and a differential manifold such that the product $(g, h) \mapsto gh$ and the inversion $g \mapsto g^{-1}$ are smooth maps.

Examples.

⋆ In Geometry: If M is a diff. manifold, the algebra of vector fields $\mathfrak{X}(M)$

⋆ In Analysis: stability group of a Pfaffian system, symmetries of solution spaces of PDEs...

⋆ In Physics: used extensively in quantum mechanics and particle physics...

The founder of Lie group theory

Sophus Lie

Sophus Lie (1842-1899) discovered that continuous transformation groups (*Lie groups*) could be better understood by "linearizing" them, and studying the related generating vector fields. They are subject to a linearized version of the group law (*commutator bracket*) and have the structure of what is today called a Lie algebra.

Algebraic inspiration:

Each algebraic equation is related to a group (Galois group, which permutes the roots) in such a way that the equation can be solved by radicals when the group is solvable!

Evariste Galois

(1811-1832)

Can we relate to any differential equation a *differential Galois group* such that both solvabilities are equivalent?

Main Correspondence Lie group - Lie algebra

Simply connected Lie groups $\longleftrightarrow$ Lie algebras (over $\mathbb{R}$)

$$G \mapsto \{X \in \mathfrak{X}(G) : (dL_g)_h(X_h) = X_{gh}\} \cong T_eG$$

$$f : G \to H \mapsto (df)_e : T_eG \to T_eH$$

Also subgroups $\leftrightarrow$ subalgebras, normal subgroups $\leftrightarrow$ ideals, etc

$\leftarrow$: As $\mathfrak{g} \leq \mathrm{gl}(n, \mathbb{R}) = (\mathrm{Mat}_{n\times n}(\mathbb{R}), [\ ,\ ])$, $G =_{gr} < \exp(\mathfrak{g}) >$

$\rightarrow$: For $G \leq \mathrm{GL}(n, \mathbb{R})$, $\mathfrak{g} = \{X \in \mathrm{gl}(n, \mathbb{R}) : e^{tX} \in G\ \forall t \in \mathbb{R}\}$

Simple examples:

- Orthogonal group (preserving a metric)

$$\rightsquigarrow \mathfrak{so}(n, \mathbb{F}) = \{A \in \mathrm{gl}(n, \mathbb{F}) : A + A^t = 0\}$$

- Special linear group (preserving a volume form)

$$\rightsquigarrow \mathfrak{sl}(n, \mathbb{F}) = \{A \in \mathrm{gl}(n, \mathbb{F}) : \mathrm{tr}(A) = 0\}$$

- Symplectic group (preserving a symplectic form)

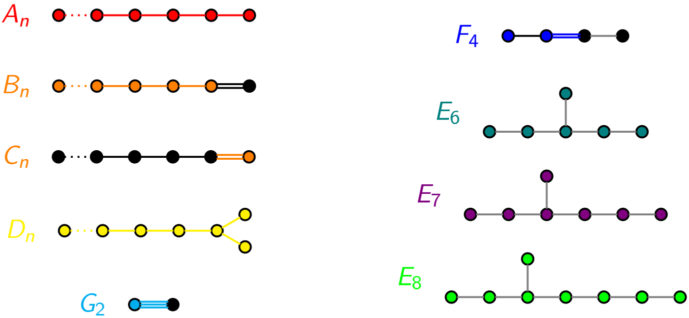
$$\rightsquigarrow \mathfrak{sp}(n, \mathbb{F}) = \{A \in \mathrm{gl}(2n, \mathbb{F}) : AC + CA^t = 0\},\ C = \begin{pmatrix} o & I_n \\ -I_n & 0 \end{pmatrix}$$

Killing 1887: Classification of f-d.simple Lie $\mathbb{C}$-alg.

Wilhelm Killing

"The greatest mathematical paper of all time"

- ⋆ $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$
- ⋆ A SURPRISE: of dimension 14, called $\mathfrak{g}_2$
- ⋆ Four additional *exceptional* examples:
  $\mathfrak{f}_4$ (52), $\mathfrak{e}_6$ (78), $\mathfrak{e}_7$ (133), $\mathfrak{e}_8$ (248).



Nilpotent Lie algebras

Definition. For $\mathfrak{g}$ a Lie algebra, the derived series/lower central series are defined as

$$\mathfrak{g}^{(n+1)} = [\mathfrak{g}^{(n)}, \mathfrak{g}^{(n)}], \quad \mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^{n}].$$

- ⋆ $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(n)} = 0$ for some n;
- ⋆ $\mathfrak{g}$ is nilpotent if $\mathfrak{g}^{n} = 0$ for some n.

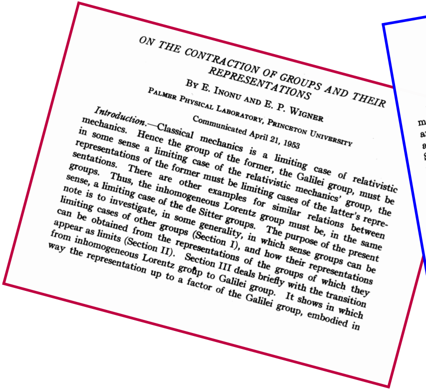
What's known:

- $\{\text{Abelian}\} \subsetneq \{\text{Nilpotent}\} \subsetneq \{\text{Solvable}\}$
- Classification in low dimensions (nilpotents up to dimension 7?)
- Levi decomposition: $\mathfrak{g}$ fin-dim $\Rightarrow \mathfrak{g} = \mathrm{Rad}(\mathfrak{g}) \rtimes$ semisimple.

Our aim

To find new families of solvable/nilpotent Lie algebras by *deforming* the (simple) exceptional ones

Motivation from Physics



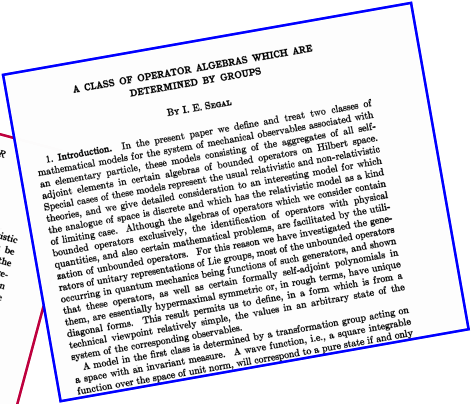
ON THE CONTRACTION OF GROUPS AND THEIR REPRESENTATIONS

BY E. INONU AND E. P. WIGNER

PALMER PHYSICAL LABORATORY, PRINCETON UNIVERSITY

Communicated April 21, 1953

Introduction.—Classical mechanics is a limiting case of relativistic mechanics. Hence the group of the former, the Galilei group, must be in some sense a limiting case of the relativistic mechanics' group, the representations of the former must be limiting cases of the latter's representations. There are other examples for similar relations between groups. Thus, the inhomogeneous Lorentz group must be, in the same sense, a limiting case of the de Sitter groups. The purpose of the present note is to investigate, in some generality, in which sense groups can be limiting cases of other groups (Section I), and how their representations can be obtained from the representations of the groups of which they appear as limits (Section II). Section III deals briefly with the transition from inhomogeneous Lorentz group to Galilei group. It shows in which way the representation up to a factor of the Galilei group, embodied in



A CLASS OF OPERATOR ALGEBRAS WHICH ARE DETERMINED BY GROUPS

BY I. E. SEGAL

1. **Introduction.** In the present paper we define and treat two classes of mathematical models for the system of mechanical observables associated with an elementary particle, these models consisting of the aggregates of all self-adjoint elements in certain algebras of bounded operators on Hilbert space. Special cases of these models represent the usual relativistic and non-relativistic theories, and we give detailed consideration to an interesting model for which the analogue of space is discrete and which has the relativistic model as a kind of limiting case. Although the algebras of operators which we consider contain bounded operators exclusively, the identification of operators with physical quantities, and also certain mathematical problems, are facilitated by the utilization of unbounded operators. For this reason we have investigated the generators of unitary representations of Lie groups, most of the unbounded operators occurring in quantum mechanics being functions of such generators, and shown that these operators, as well as certain formally self-adjoint polynomials in them, are essentially hypermaximal symmetric or, in rough terms, have unique diagonal forms. This result permits us to define, in a form which is from a technical viewpoint relatively simple, the values in an arbitrary state of the system of the corresponding observables.

A model in the first class is determined by a transformation group acting on a space with an invariant measure. A wave function, i.e., a square integrable function over the space of unit norm, will correspond to a pure state if and only

Inonu and Wigner, 1953:
Galilei group limiting case of the relativistic mechanics group

Segal, 1951:
Sequence of groups whose structure constants converge toward the structure constants of a non-isomorphic group

A world of concepts:

Continuous contractions // Degenerations // Graded contractions

Continuous contractions

Def. $\mathbb{F}$ field, $\mathcal{L}$ Lie $\mathbb{F}$-algebra.
If $U\colon (0,1] \to \mathrm{GL}(\mathcal{L})$, $\varepsilon \in (0,1] \mapsto U_\varepsilon$, we define

$$[x,y]_\varepsilon = U_\varepsilon^{-1}([U_\varepsilon(x), U_\varepsilon(y)]).$$

Note $\mathcal{L}_\varepsilon := (\mathcal{L}, [\ ,\ ]_\varepsilon) \cong \mathcal{L}$.
Assume for any x, y there exists $\lim_{\varepsilon\to 0}[x,y]_\varepsilon (=: [x,y]_0)$

$\Rightarrow \mathcal{L}_0 := (\mathcal{L}, [\ ,\ ]_0)$ is a Lie algebra called
one-parametric continuous contraction of $\mathcal{L}$.

Example

$$\mathcal{L} = \mathfrak{so}(3) = \left\langle e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

Take $U_\varepsilon : e_1 \mapsto \varepsilon e_1,\ e_2 \mapsto \varepsilon e_2,\ e_3 \mapsto e_3$

$$\left.\begin{aligned} [e_1,e_2]_\varepsilon &= \varepsilon^2 e_3 \\ [e_2,e_3]_\varepsilon &= e_1 \\ [e_3,e_1]_\varepsilon &= e_2 \end{aligned}\right\} \Rightarrow \mathcal{L}_\varepsilon \cong \mathcal{L} \quad \text{but} \quad \left.\begin{aligned} [e_1,e_2]_0 &= 0 \\ [e_2,e_3]_0 &= e_1 \\ [e_3,e_1]_0 &= e_2 \end{aligned}\right\} \Rightarrow \mathcal{L}_0 \text{ solvable}$$

Degenerations

$\mathbb{F}$ arbitrary field, V $\mathbb{F}$-vector space of dimension n

Consider the variety of Lie algebras on V:

$$\mathcal{L}_n(V) = \{\mu \colon V \times V \to V : (V,\mu) \text{ Lie algebra}\}$$

$$\equiv \{\mu \in V^* \otimes V^* \otimes V : (V,\mu) \text{ Lie algebra}\} \text{ subvariety of } V^* \otimes V^* \otimes V$$

$$\equiv \mathcal{C}_n(\mathbb{F}) = \left\{ (c_{ij}^k) \in \mathbb{F}^{n^3} : \begin{array}{l} 0 = c_{ij}^k + c_{ji}^k \\ 0 = \sum_{r=1}^n (c_{ij}^r c_{kr}^s + c_{jk}^r c_{ir}^s + c_{ki}^r c_{jr}^s) \end{array} \right\}$$

$\mathrm{GL}(V)$ acts on $\mathcal{L}_n(V)$: $\quad g \cdot \mu(x,y) := g(\mu(g^{-1}x, g^{-1}y))$

$\rightsquigarrow$ orbit $O(\mu) = \{g \cdot \mu : g \in \mathrm{GL}(V)\}$

Def. $\mu, \lambda \in \mathcal{L}_n(V)$. μ degenerates to λ if $\lambda \in \overline{O(\mu)}$

(clausure in the Zariski topology)

- λ a degeneration of μ is trivial if $\mu \approx \lambda$
- μ is rigid if $O(\mu)$ is open in $\mathcal{L}_n(V)$

Gradings

For graded contractions we need a grading:

G abelian group, $\mathcal{L}$ Lie algebra over $\mathbb{F}$,

Def. $\Gamma : \mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ is a G-grading on $\mathcal{L}$ if $[\mathcal{L}_g, \mathcal{L}_h] \subset \mathcal{L}_{g+h} \quad \forall g, h \in G$.

Example. On $\mathcal{L} = \mathfrak{sl}_2(\mathbb{F})$:

$$G = \mathbb{Z},\ \mathcal{L}_0 = \langle \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{h} \rangle,\ \mathcal{L}_1 = \langle \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{e} \rangle,\ \mathcal{L}_{-1} = \langle \underbrace{\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}}_{f} \rangle$$

$$G = \mathbb{Z}_2^2,\ \mathcal{L}_{(\bar{1},\bar{0})} = \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rangle,\ \mathcal{L}_{(\bar{1},\bar{1})} = \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle,\ \mathcal{L}_{(\bar{0},\bar{1})} = \langle \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rangle$$

Graded contractions

G abelian group, $\Gamma : \mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ a Lie algebra over $\mathbb{F}$.

Def. A graded contraction of Γ is a map $\varepsilon\colon G\times G\to\mathbb{F}$
such that $\mathcal{L}^\varepsilon = (\mathcal{L}, [\ ,\ ]^\varepsilon)$ is a Lie algebra,
where $[x,y]^\varepsilon := \varepsilon(g,h)[x,y]$ if $x\in\mathcal{L}_g,\ y\in\mathcal{L}_h,\ g,h\in G$,

Two *opposite* examples:

$\varepsilon(g,h)=1\ \forall g,h \Rightarrow \mathcal{L}^\varepsilon=\mathcal{L}$
$\varepsilon(g,h)=0\ \forall g,h \Rightarrow \mathcal{L}^\varepsilon$ abelian

$\rightarrow$ Source for finding solvable and nilpotent Lie algebras

Example:
$\mathcal{L}=\mathfrak{sl}_2(\mathbb{R})\not\cong\mathfrak{so}(3)=\mathcal{M}$

but we can pass from $\mathcal{L}$ to $\mathcal{M}$ by a graded contraction:

$\mathcal{L}^\varepsilon$	$\xrightarrow{\cong}$	$\mathcal{M}$
h	$\mapsto$	$-e_2$
e	$\mapsto$	e_3
f	$\mapsto$	e_1

for $\varepsilon\colon\mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$
$\varepsilon(-1,1)=\varepsilon(1,-1)=1$
$\varepsilon(0,n)=\varepsilon(n,0)=0$

Problem of classification

Given $\Gamma : \mathcal{L} = \oplus_{g \in G} \mathcal{L}_g$ a G-grading,

how many different Lie algebras
can be obtained by a graded contraction?

Def. If ε, ε' are graded contractions of Γ, ε is equivalent to ε' ($\varepsilon \sim \varepsilon'$) if
$\exists \varphi \colon \mathcal{L}^{\varepsilon} \to \mathcal{L}^{\varepsilon'}$ (graded) isomorphism of Lie algebras

General AIM: to classify $\{$graded contractions of $\Gamma\}/\sim$

Def. ε is equivalent by normalization to ε' ($\varepsilon \sim_n \varepsilon'$) if
$\exists \varphi \colon \mathcal{L}^{\varepsilon} \to \mathcal{L}^{\varepsilon'}$ isomorphism of graded Lie algebras with $\varphi|_{\mathcal{L}_g} = \alpha_g \operatorname{id}$

Remark: of course $\varepsilon \sim_n \varepsilon' \Rightarrow \varepsilon \sim \varepsilon'$, but $\sim_n$-examples are easy to obtain:

$$\text{given } \varepsilon \text{ and } \alpha \colon G \to \mathbb{F}^{\times} \Rightarrow \left\{ \begin{array}{l} \varepsilon^{\alpha} \colon G \times G \to \mathbb{F}, \\ \varepsilon^{\alpha}(g,h) = \varepsilon(g,h) \frac{\alpha(g)\alpha(h)}{\alpha(g+h)} \end{array} \right\} \sim_n \varepsilon$$

How a graded contraction is?

Given a map $\varepsilon\colon G \times G \to \mathbb{F}$, and $\Gamma : \mathcal{L} = \oplus_{g\in G}\mathcal{L}_g$ a G-grading on $\mathcal{L}$,

Which condition has ε to satisfy to be a graded contraction?

Easy answer:

- $[\ ,\ ]_\varepsilon$ skew-symmetric $\Leftrightarrow (\varepsilon(g,h) - \varepsilon(h,g))[x,y] = 0$
- $[\ ,\ ]_\varepsilon$ satisfies Jacobi identity $\Leftrightarrow \forall k,g,h \in G, x \in \mathcal{L}_g, y \in \mathcal{L}_h, z \in \mathcal{L}_k$,

$$(\varepsilon(g,h,k) - \varepsilon(k,g,h))[x,[y,z]] + (\varepsilon(h,k,g) - \varepsilon(k,g,h))[y,[z,x]] = 0$$

where $\varepsilon(g,h,k) := \varepsilon(g,h+k)\varepsilon(h,k)$

Enough conditions: $\forall g,h,k \in G$,

- ⋆ $\varepsilon(g,h) = \varepsilon(h,g)$
- ⋆ $\varepsilon(g,h,k) = \varepsilon(k,g,h)$

not necessary! (in general)

$\Rightarrow$ the study of the graded contractions depends strongly on Γ!

Earlier works on graded contractions

Some precedents in literature:

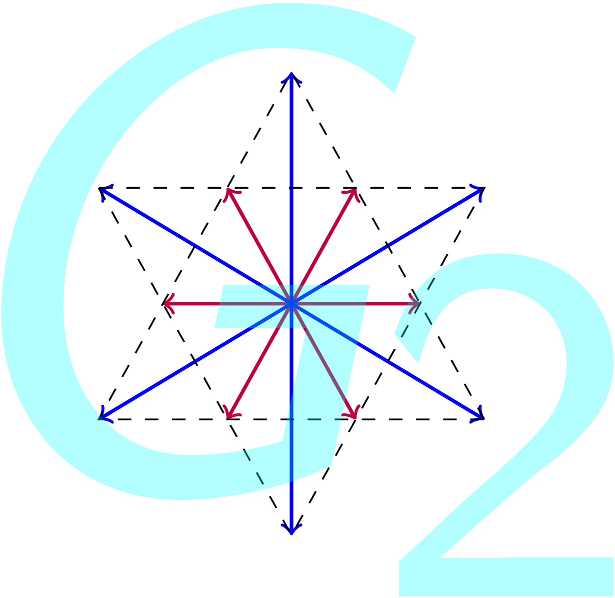
* de Montigny, Patera: J. Phys. A 1991: first work ($\mathbb{Z}_2$-gradings)
* Couture, Patera, Sharp, Winternitz: JMP 1991: $\mathfrak{sl}(3,\mathbb{C})$
* Hrivnák, Novotný, Patera, Tolar, LAA 2006: $\mathfrak{sl}(3,\mathbb{C})$, $\mathbb{Z}_3^2$-grad (Pauli)
* Hrivnák, Novotný, JMP 2013: $\mathfrak{sl}(3,\mathbb{C})$, $\mathbb{Z}_2^3$-grading (Gell-Mann)
* Weimar-Woods, Can. J. Math. 2006: general structure
* Escobar, Núñez, Pérez-Fernández, 2018: filiform Lie algebras

Our (first) aim:

The 14-dimensional exceptional Lie algebra $\mathfrak{g}_2$
endowed with a grading in 2-dimensional pieces

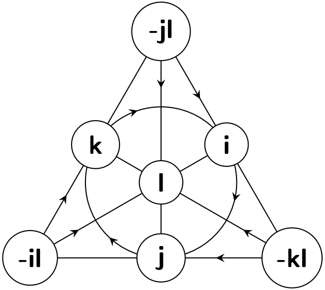
WITHOUT COMPUTER!

What is $\mathfrak{g}_2$?



Octonion algebra

Cayley-Dickson doubling process: $\mathbb{F} \xrightarrow{\mathbb{Z}_2} \mathbb{F} \oplus \mathbb{F}\mathbf{i} \xrightarrow{\mathbb{Z}_2} \mathbb{H} \xrightarrow{\mathbb{Z}_2} \mathbb{O}$



$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\mathbf{l}$ is $\mathbb{Z}_2^3$-graded:

$$\underbrace{\begin{array}{l} \mathbb{O}_{(000)} = \mathbb{F}1 \\ \mathbb{O}_{(100)} = \mathbb{F}\mathbf{i} \\ \mathbb{O}_{(010)} = \mathbb{F}\mathbf{j} \\ \mathbb{O}_{(110)} = \mathbb{F}\mathbf{k} \end{array}}_{\mathbb{H}} \qquad \underbrace{\begin{array}{l} \mathbb{O}_{(001)} = \mathbb{F}\mathbf{l} \\ \mathbb{O}_{(101)} = \mathbb{F}\mathbf{il} \\ \mathbb{O}_{(011)} = \mathbb{F}\mathbf{jl} \\ \mathbb{O}_{(111)} = \mathbb{F}\mathbf{kl} \end{array}}_{\mathbb{H}\mathbf{l}}$$

$(\mathrm{Der}(\mathbb{O}), [\ ,\ ])$ is a simple Lie algebra of dimension 14 of type G_2!

$\mathfrak{g}_2$ AND ITS $\mathbb{Z}_2^3$-GRADING

$\mathcal{L} = \text{Der}(\mathbb{O}) = \{d\colon \mathbb{O} \to \mathbb{O} \text{ lin} : d(xy) = d(x)y + xd(y)\ \forall x, y \in \mathbb{O}\}$
is a simple Lie algebra of dimension 14 of type G_2

As $\mathbb{O}$ is $\mathbb{Z}_2^3$-graded $\Rightarrow$ $\Gamma_{\mathfrak{g}_2} : \mathcal{L} = Der(\mathbb{O}) = \oplus_{g \in \mathbb{Z}_2^3} \mathcal{L}_g$ is $\mathbb{Z}_2^3$-graded too:

$$\mathcal{L}_g = \{d \in \text{Der}(\mathbb{O}) : d(\mathbb{O}_h) \subset \mathbb{O}_{g+h}\ \forall h \in \mathbb{Z}_2^3\}$$

Main features of this grading

- Fine grading (it has no proper refinements)
- Non-toral grading (not compatible with any root decomposition)

How its homogeneous components are?

- ⋆ $\mathcal{L}_e = 0$,
- ⋆ $\dim \mathcal{L}_g = 2$ for all $e \neq g \in \mathbb{Z}_2^3$: each $\mathcal{L}_g$ is a Cartan subalgebra

$\Rightarrow$ Any homogeneous element is semisimple

$\rightsquigarrow$ This $\Gamma_{\mathfrak{g}_2}$ is the grading we are going to contract

Graded contractions of $\Gamma_{\mathfrak{g}_2}$

AIM: To classify graded contractions of $\Gamma_{\mathfrak{g}_2}$ up to $\sim$

As $\mathcal{L}_e = 0 \Rightarrow [\mathcal{L}_g, \mathcal{L}_g] = [\mathcal{L}_g, \mathcal{L}_e] = [\mathcal{L}_e, \mathcal{L}_g] = 0 \quad \forall g \in \mathbb{Z}_2^3 \rightsquigarrow$

Def. $\varepsilon \colon G \times G \to \mathbb{F}$ is said admissible if $\varepsilon(g,g) = \varepsilon(g,e) = \varepsilon(e,g) = 0$

Not every graded contraction is admissible but

Lemma. If $\varepsilon \colon G \times G \to \mathbb{F}$ is a graded contraction of $\Gamma_{\mathfrak{g}_2}$,

$\Rightarrow \exists\ \varepsilon'$ admissible graded contraction of $\Gamma_{\mathfrak{g}_2}$ equivalent to ε.

More properties of $\Gamma_{\mathfrak{g}_2}$ relevant for our approach

(P1) $[\mathcal{L}_g, \mathcal{L}_h] = \mathcal{L}_{g+h}$ if $g, h, g+h \neq e$;

(P2) If $\langle g, h, k \rangle = \mathbb{Z}_2^3 \Rightarrow \exists x \in \mathcal{L}_g, y \in \mathcal{L}_h, z \in \mathcal{L}_k$ such that $\{[x,[y,z]], [y,[z,x]]\}$ linearly independent set

Consequence: Fixed $\varepsilon \colon G \times G \to \mathbb{F}$ admissible map,

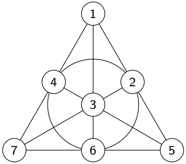
$$\varepsilon \text{ graded contraction} \Leftrightarrow \begin{cases} \varepsilon(g,h) = \varepsilon(h,g) \\ \varepsilon(g,h,k) = \varepsilon(h,k,g) \text{ if } \langle g,h,k \rangle = \mathbb{Z}_2^3 \end{cases}$$

$\rightsquigarrow$ We can forget of the grading $\Gamma_{\mathfrak{g}_2}$ and think only of the grading group $\mathbb{Z}_2^3$

Towards a combinatorial approach

Cleaning a little bit

In admissible graded contractions of $\Gamma_{\mathfrak{g}_2}$ the only important thing is the image of a pair $\{g,h\}$ with $g \neq h \neq e$, so:



- ⋆ Forget octonions in Fano plane
- ⋆ Think only of the indices $I = \{1,2,3,4,5,6,7\}$
- ⋆ $i * j \in I$ is (partially) defined by $g_{i*j} = g_i + g_j$:

$$\begin{array}{llll} g_0 = (\bar{0},\bar{0},\bar{0}) & g_1 = (\bar{1},\bar{0},\bar{0}) & g_2 = (\bar{0},\bar{1},\bar{0}) & g_3 = (\bar{0},\bar{0},\bar{1}) \\ g_4 = (\bar{1},\bar{1},\bar{1}) & g_5 = (\bar{1},\bar{1},\bar{0}) & g_6 = (\bar{1},\bar{0},\bar{1}) & g_7 = (\bar{0},\bar{1},\bar{1}) \end{array}$$

- ⋆ We call $\{ijk\}$ a generating triplet if $\langle g_i, g_j, g_k \rangle = \mathbb{Z}_2^3$
- ⋆ $X = \{\{i,j\} : i \neq j, i,j \in I\}$ 21 elements

$$\left\{ \begin{array}{l} \text{admissible graded} \\ \text{contractions of } \Gamma_{\mathfrak{g}_2} \end{array} \right\} \xrightarrow{1-1} \mathcal{A} = \left\{ \eta\colon X \to \mathbb{F} : \begin{array}{l} \eta_{ijk} = \eta_{jki} \\ \forall \{ijk\} \text{ generating triplet} \end{array} \right\}$$

$$\begin{array}{ccccc} \varepsilon & \mapsto & \eta^{\varepsilon}: & X & \to & \mathbb{F} \\ & & & \{i,j\} & \mapsto & \eta^{\varepsilon}_{ij} := \varepsilon(g_i, g_j) \end{array}$$

Notation: $\eta_{ijk} := \eta_{i\,j*k}\eta_{jk}$

How to find elements in $\mathcal{A}$?

Recall: $\mathcal{A} = \{\eta \colon X \to \mathbb{F} : \eta_{ijk} = \eta_{jki} \; \forall \{ijk\} \text{ generating triplet}\}$

Example of an element in $\mathcal{A}$:

$\eta \colon X =$

12	13	14	15	16	17 $\to \mathbb{F}$
	23	24	25	26	27
		34	35	36	37
			45	46	47
				56	57
					67

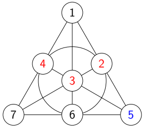
$* \mapsto 1$ (red: 23, 24, 25, 34, 36, 47)

$* \mapsto 0$ (blue: all others)

What is what we need to find examples? $\sup(\eta) := \{t \in X : \eta(t) \neq 0\}$

$$\text{If some } \eta_{ijk} \neq 0 \Longrightarrow \eta_{jki} \neq 0 \text{ and } \eta_{kij} \neq 0$$

Example: If $\underbrace{\underbrace{\eta_{234}}_{\eta_{2\,3*4}\eta_{34}}}_{34,25 \in \sup(\eta)} \neq 0 \Longrightarrow \underbrace{\underbrace{\eta_{342}}_{\eta_{3\,4*2}\eta_{42}}}_{36,42 \in \sup(\eta)} \neq 0$ and $\underbrace{\underbrace{\eta_{423}}_{\eta_{4\,2*3}\eta_{23}}}_{47,23 \in \sup(\eta)} \neq 0$:



$\rightsquigarrow$ the support is not arbitrary: it satisfies a kind of absorbing property

Nice sets

If $\{i, j, k\}$ is a generating triplet, take

$$P_{ijk} := \big\{\{i,j\},\{j,k\},\{k,i\},\{i,j*k\},\{j,k*i\},\{k,i*j\}\big\} \subset X.$$

Def. $T \subset X$ is said a nice set if

whenever $\{j,k\},\{i,j*k\} \in T$ then $P_{ijk} \subset T$.

Proposition

- ⋆ If $\eta \in \mathcal{A}$, the support of η is a nice set;
- ⋆ For any nice set T, the map $\eta^T \in \mathcal{A}$ for $\eta^T : \left.\begin{array}{ccc} X & \to & \mathbb{F} \\ t \in T & \mapsto & 1 \\ t \notin T & \mapsto & 0 \end{array}\right\}$

⇝ next aim: to classify nice sets

Collineations

Given a grading Γ on a Lie algebra $\mathcal{L}$:

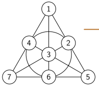
- $\mathrm{Aut}(\Gamma) = \{f \in \mathrm{Aut}(\mathcal{L}) : \forall g \in G \text{ there is } g' \text{ with } f(\mathcal{L}_g) \subseteq \mathcal{L}_{g'}\}$.
- $\mathrm{Stab}(\Gamma) = \{f \in \mathrm{Aut}(\mathcal{L}) : f(\mathcal{L}_g) \subseteq \mathcal{L}_g \ \forall\, g \in \mathcal{L}\}$.
- The Weyl group of Γ is the quotient group $\mathcal{W}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Stab}(\Gamma)$.

It reduces the quadratic system of equations which gives the graded contractions

---

Our case. $\mathcal{W}(\Gamma_{\mathfrak{g}_2}) \cong \mathrm{Aut}(\mathbb{Z}_2^3) = \mathrm{Gl}(3, \mathbb{Z}_2) \cong \mathrm{Coll}\, I$:

Def. A bijection $\sigma\colon I \to I$ is said to be a collineation
if it applies lines to lines, i.e., $\sigma(i * j) = \sigma(i) * \sigma(j)$.



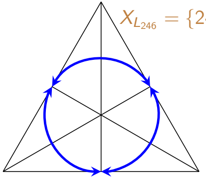
$\to$ Example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 3 & 2 & 7 & 4 & 1 \end{pmatrix}$ fixes the set of 7 lines
$\{(125), (567), (741), (136), (642), (273), (345)\}$

So we have an action
$$\begin{array}{rcl} \mathrm{Coll}\, I \times \mathcal{A} & \to & \mathcal{A} \\ (\sigma, \eta) & \mapsto & \sigma \cdot \eta\colon X \to \mathbb{F} \\ & & \qquad \{ij\} \mapsto \eta_{\sigma(i)\sigma(j)} \end{array}$$
and $\sigma \cdot \eta \sim \eta$

Main examples of nice sets

If L is a line,

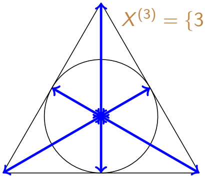
$$X_L = \{t \in X : t \subset L\}$$



$X_{L_{246}} = \{24, 46, 62\}$

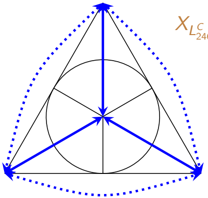
If i is an index,

$$X_{(i)} = \{t \in X : i \in t\}$$



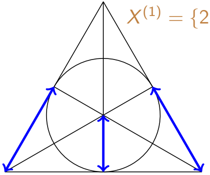
$X^{(3)} = \{31, 32, 34, 35, 36, 37\}$

$$X_{L^c} = \{t \in X : t \subset L^c\}$$



$X_{L^C_{246}} = \left\{ \begin{matrix} 31, 35, 37 \\ 15, 57, 71 \end{matrix} \right\}$

$$X^{(i)} = \{\{jk\} \in X : j * k = i\}$$



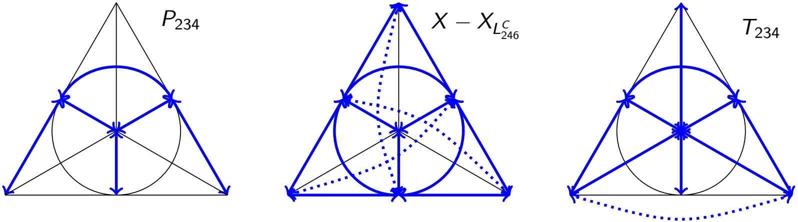
$X^{(1)} = \{25, 36, 47\}$

All their subsets are nice sets too

Remaining nice sets

$\emptyset$, X and

- $\star$ $P_{ijk} = \{ij, jk, ki, i\,j*k, j\,k*i, k\,i*j\}$ if $\{ijk\}$ generating triplet,
- $\star$ $X \setminus X_{L^c}$
- $\star$ $T_{ijk} = P_{ijk} \cup P_{ij\,i*k} \cup P_{ik\,i*j} \cup P_{i\,i*j\,i*k}$
  $= \{ij, jk, ki, i\,j*k, i\,i*j, i\,i*k, i\,i*j*k, i*j\,i*k, k\,i*j, j\,i*k\}$



Not more nice sets (up to collineations)
Proof: only combinatorial

ONE ALGEBRA FOR EACH SUPPORT?? Until now:

Hence there are 24 nice sets up to collineation:

Cardinal	0	1	2	3	4	5	6	10	15	21	
How many	1	1	3	7	4	2	3	1	1	1	24

We have:

- ⋆ All the possible supports (up to collineations)
- ⋆ At least one Lie algebra for any nice set

  $\Rightarrow$ At least 22 not isomorphic algebras not simple and not abelian

Exactly one algebra for each support? Not necessarily

Example $T = \{\{1,2\},\{1,3\},\{1,5\},\{1,6\}\}$.

- ⋆ Write $\eta = (\eta_{12}, \eta_{13}, \eta_{15}, \eta_{16})$ (recall $\eta_{ij} = 0$ if $\{i,j\} \notin T$)
- ⋆ Any of them belongs to $\mathcal{A}$ $\rightsquigarrow$ provides a graded contraction

  $\rightsquigarrow$ provides a related Lie algebra $\mathcal{L}^{\eta}$.
- ⋆ For instance $\eta_1 = (1,1,1,1)$, $\eta_2 = (2,1,1,1)$ and $\eta_3 = (1,1,1,2)$:

$$\mathcal{L}^{\eta_1} \cong \mathcal{L}^{\eta_2} \ncong \mathcal{L}^{\eta_3}$$

Orbits up to normalization: complex case

(Normalization preserves supports, conversely?)

Example: $T = X_{L_{125}} = \{\{1,2\},\{1,5\},\{2,5\}\}$.

- ⋆ Take $\eta\colon X \to \mathbb{C}$ with support T.
- ⋆ Write $\eta = (\eta_{12}, \eta_{15}, \eta_{25}) \in (\mathbb{C}^{\times})^3$ (recall $\eta_{ij} = 0$ if $\{i,j\} \notin T$)
- ⋆ Recall $\eta \sim_n \eta' \iff \exists \alpha\colon I \to \mathbb{C}^{\times}$ such that $\frac{\eta_{ij}}{\eta'_{ij}} = \frac{\alpha_i \alpha_j}{\alpha_{i*j}}$
- ⋆ Hence $\eta \sim_n (1,1,1) \iff \exists \alpha\colon I \to \mathbb{C}^{\times}$ such that $\begin{cases} \frac{\alpha_1\alpha_2}{\alpha_5} = \eta_{12} \\ \frac{\alpha_1\alpha_5}{\alpha_2} = \eta_{15} \\ \frac{\alpha_2\alpha_5}{\alpha_1} = \eta_{25} \end{cases}$
- ⋆ This system has solution in $\mathbb{C}$; for instance $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_7 = 1$ and:

$$\alpha_1 = \sqrt{\eta_{12}}\sqrt{\eta_{15}}, \quad \alpha_2 = \sqrt{\eta_{12}}\sqrt{\eta_{25}}, \quad \alpha_5 = \sqrt{\eta_{15}}\sqrt{\eta_{25}}$$

- ⋆ Only one graded Lie algebra obtained here:

$$\mathcal{L}^{\varepsilon} = \underbrace{(\mathcal{L}_{g_1} \oplus \mathcal{L}_{g_2} \oplus \mathcal{L}_{g_5})}_{\substack{[\mathcal{L},\mathcal{L}]^{\varepsilon} \cong 2\mathfrak{sl}(2,\mathbb{C}) \\ \text{semisimple}}} \oplus \underbrace{(\mathcal{L}_{g_2} \oplus \mathcal{L}_{g_4} \oplus \mathcal{L}_{g_6} \oplus \mathcal{L}_{g_7})}_{\substack{Z(\mathcal{L}^{\varepsilon}) = \mathrm{Rad}(\mathcal{L}^{\varepsilon}) \\ \text{centre}}}$$

Results on orbits up to normalization

Order $\sup(\eta) = \{t_1, \dots, t_s\}$ lexicographically.
Write η by $(\eta(t_1), \dots, \eta(t_s))$.

Theorem.

Let T be a nice set and $\eta \in \mathcal{A}$ such that $\sup(\eta) = T$.

a) If $T = \{\{1,2\},\{1,3\},\{1,5\},\{1,6\}\} \Rightarrow \eta \sim_n (1,1,1,\lambda)$, and
$$(1,1,1,\lambda) \sim_n (1,1,1,\lambda') \Leftrightarrow \lambda = \lambda'.$$

b) If $T = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\}\} \Rightarrow \eta \sim_n (1,\lambda,1,1,\lambda)$, and
$$(1,\lambda,1,1,\lambda) \sim_n (1,\lambda',1,1,\lambda') \Leftrightarrow \lambda = \pm\lambda'.$$

c) If $T = X_{(1)} \Rightarrow \eta \sim_n (1,\lambda,\mu,1,\lambda,\mu)$, and
$$(1,\lambda,\mu,1,\lambda,\mu) \sim_n (1,\lambda',\mu',1,\lambda',\mu') \Leftrightarrow \lambda = \pm\lambda',\ \mu = \pm\mu'.$$

d) Otherwise, $\eta \sim_n (1,\dots,1)$.

Hence, we have 21 graded Lie algebras of dimension 14 obtained by contracting the $\mathbb{Z}_2^3$-grading on $\mathfrak{g}_2$, jointly with 3 families depending or one or two parameters... Could be these Lie algebras isomorphic? Could...

EQUIVALENCE CLASSES A posteriori, they are not isomorphic

Theorem Not more!

That is, $\eta \sim \eta' \Leftrightarrow \eta \sim_n \eta'$

So, we have really obtained the classification of the graded contractions of $\Gamma_{\mathfrak{g}_2}$
(no precedents in this part)

Main tool of the proof: Some specific facts on $\Gamma_{\mathfrak{g}_2}$: for each line

$$\begin{array}{ccccc} \mathcal{L}_i & \oplus & \mathcal{L}_j & \oplus & \mathcal{L}_{i*j} = \text{two copies of } \mathfrak{so}(3,\mathbb{C}) = \langle e_1, e_2, e_3 \rangle \\ e_1 & & e_2 & & e_3 \\ e_1' & & e_2' & & e_3' \end{array}$$

$\rightsquigarrow$ a graded isomorphism perhaps is not a scalar α_i id in $\mathcal{L}_i$,
but can be treated as an endomorphism of
a 2-dimensional vector space with properties

Properties of the obtained Lie algebras

21 isolated and 3 infinite families , all of dimension 14

More abelian

- One is abelian;
- 13 Nilpotent (not abelian): one with nilindex 3 and the remaining ones with nilindex 2;
- "7"Solvable (not nilpotent):
    - A one-parametric family (of continuum graded Lie algebras) 2-step solvable, $\dim \mathfrak{z}(\mathcal{L}_\varepsilon) = 4$, $\dim[\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon] = 8$,
    - A one-parametric family of 2-step solvable Lie algebras, $\dim \mathfrak{z}(\mathcal{L}_\varepsilon) = 2$, $\dim[\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon] = 10$,
    - A two-parametric family of 2-step solvable Lie algebras, $\dim \mathfrak{z}(\mathcal{L}_\varepsilon) = 0$, $\dim[\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon] = 12$,
    - two with solvability index 3,
    - two more with solvability index 2;
- One is sum of a semisimple Lie algebra with a 8-dim center;
- One is not reductive (without center);
- One is simple.

Less abelian

Other "exceptional" Lie algebras

$\mathbb{Z}_2^3$-graded Lie algebras related to Octonions - First part

$$\left.\begin{array}{lll} \mathcal{L} & = \mathfrak{g}_2 = \mathfrak{der}(\mathbb{O}) \\ \cap & \\ \mathcal{M} & = \mathfrak{b}_3 = \mathfrak{der}(\mathbb{O}) \oplus \mathrm{ad}(\mathbb{O}_0) \\ \cap & \\ \mathcal{N} & = \mathfrak{d}_4 = \mathfrak{der}(\mathbb{O}) \oplus L_{\mathbb{O}_0} \oplus R_{\mathbb{O}_0} \end{array}\right\}$$

$$\begin{array}{r} \mathbb{O} \to \mathbb{O} \\ L_x(y) = xy \\ R_x(y) = yx \\ \mathrm{ad}_x = L_x - R_x \end{array}$$

$\mathbb{Z}_2^3$-grading on $\mathbb{O} \Rightarrow$ $\mathbb{Z}_2^3$-grading on $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{N}$. And $\forall g \neq e$:

$$\begin{array}{l|l|l} \mathcal{L}_e = 0 & \mathcal{M}_e = 0 & \mathcal{N}_e = 0 \\ \dim \mathcal{L}_g = 2 & \dim \mathcal{M}_g = 3 & \dim \mathcal{N}_g = 4 \end{array}$$

All of them P1+P2 $\Rightarrow$ Nice sets = supports of the graded contractions of $\Gamma_{\mathfrak{b}_3}$ and $\Gamma_{\mathfrak{d}_4}$

$\Longrightarrow$ 21 isolated cases + 3 families of Lie algebras of dimension 21
and 21 isolated cases + 3 families of Lie algebras of dimension 28!

Properties of the obtained algebras Case D_4

21 isolated and 3 infinite families

All have dimension 28:

- One is abelian;
- 13 Nilpotent (not abelian): one with nilindex 3 and the remaining ones with nilindex 2;
- "7"Solvable (not nilpotent):
  - A one-parametric family (of continuum graded Lie algebras) 2-step solvable, $\dim \mathfrak{z}(\mathcal{L}_\varepsilon) = 8$, $\dim[\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon] = 16$,
  - A one-parametric family of 2-step solvable Lie algebras, $\dim \mathfrak{z}(\mathcal{L}_\varepsilon) = 4$, $\dim[\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon] = 20$,
  - A two-parametric family of 2-step solvable Lie algebras, $\dim \mathfrak{z}(\mathcal{L}_\varepsilon) = 0$, $\dim[\mathcal{L}_\varepsilon, \mathcal{L}_\varepsilon] = 24$,
  - two with solvability index 3,
  - two more with solvability index 2;
- One is sum of a semisimple Lie algebra and a center of dimension 16
- One is not reductive (without center);
- One is simple.

The big exceptional Lie algebras

$\mathbb{Z}_2^3$-graded Lie algebras related to Octonions - Second part

Tits unified construction (1966)

$$\mathcal{L} = \mathfrak{der}(\mathbb{O}) \quad \oplus \quad \mathbb{O}_0 \otimes \mathcal{J}_0 \quad \oplus \quad \mathfrak{der}(\mathcal{J})$$

$\mathcal{J} = \mathbb{R} \rightsquigarrow \mathcal{L} = \mathfrak{g}_2$			14
$\mathcal{J} = \mathcal{H}_3(\mathbb{R}) \rightsquigarrow \mathcal{L} = \mathfrak{f}_4$		$\mathbb{Z}_2^3$-gradings	52
$\mathcal{J} = \mathcal{H}_3(\mathbb{C}) \rightsquigarrow \mathcal{L} = \mathfrak{e}_6$	}	directly induced from $\mathbb{O}$	78
$\mathcal{J} = \mathcal{H}_3(\mathbb{H}) \rightsquigarrow \mathcal{L} = \mathfrak{e}_7$		in all the cases	133
$\mathcal{J} = \mathcal{H}_3(\mathbb{O}) \rightsquigarrow \mathcal{L} = \mathfrak{e}_8$			248

- $\mathcal{L}_e = 0$? NO
- $\dim \mathcal{L}_g$ independent of $g (\neq e)$? YES
- Nice set are the supports again? NO

But generalized nice sets yes!!

This is another story....

Final conclusions

Some of the achievements of this work have been:

- ⋆ To avoid computer
- ⋆ We have increased the dimension
- ⋆ We have been able to continue the classification after normalization process
- ⋆ We have applied our results to a nice family of (considerably big) Lie algebras

And the work in progress in this moment is:

- ⋆ To classify generalized nice sets
- ⋆ To board the real case

Thank you for your attention!