Coprime Automorphisms of Finite Groups

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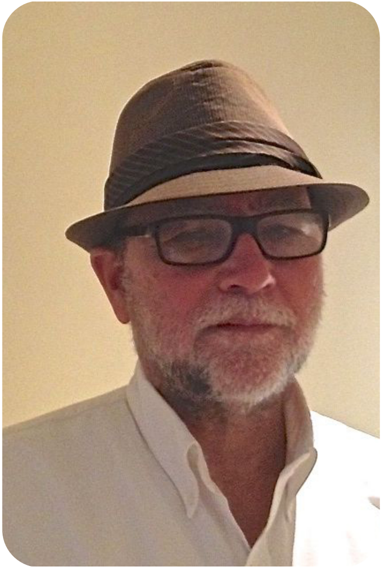
Joint work with



Robert Guralnick (USC)

Pavel Shumyatsky (UnB)

Initial settings

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If N is any α-invariant normal subgroup of G we have:

(i) $C_{G/N}(\alpha) = C_G(\alpha)N/N$, and $I_{G/N}(\alpha) = \{gN \mid g \in I_G(\alpha)\}$;

(ii) If $N = C_N(\alpha)$, then $[G, \alpha]$ centralizes N.

Influence of $C_G(\alpha)$ on G

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If G admits an automorphism α of prime order p with $C_G(\alpha)$ of order m, then G has a nilpotent subgroup of (m, p)-bounded index and p-bounded class.

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If G admits a coprime automorphism α of prime order p with $C_G(\alpha)$ of rank r, then G has characteristic subgroups $R \leq N$ such that N/R is nilpotent of p-bounded class, while R and G/N have (p, r)-bounded ranks.

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The rank of a finite group G is the least number r such that each subgroup of G can be generated by at most r elements.

Dual problem with $I_G(\alpha)$

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Note that $[G, \alpha]$ commutes with N and so $[[G, \alpha] : Z([G, \alpha])] \leq m!$.
The Schur theorem yields that $|[G, \alpha]'|$ is m-bounded.
We can pass to $G/[G, \alpha]'$ and assume that $[G, \alpha]$ is abelian.
Then $[G, \alpha] = I_G(\alpha)$ and so $|[G, \alpha]| \leq m$. □

A rank condition on the set $I_G(\alpha)$

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Theorem 1

Let G be a finite group admitting a coprime automorphism α of order e and suppose that any subgroup generated by a subset of $I_G(\alpha)$ can be generated by r elements. Then $[G, \alpha]$ has (e, r)-bounded rank.

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- for soluble groups: one key step is to show that there exists an (e, r)-bounded number f such that the fth term of the derived series of $[G, \alpha]$ is nilpotent (Zassenhaus' theorem on the derived length of any soluble subgroup of $GL_n(k)$ and Hartley-Isaacs result on representation theory).

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- the general case: after a long reduction it is sufficient to prove the result in the case where G is soluble-by-semisimple-by-soluble. It depends on CFSG, on facts about conjugacy classes and characters of $PGL_2(q)$ and also on the following result (of independent interest)

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Theorem 2

Let G be a finite group admitting a coprime automorphism α such that $g^{-1}g^{\alpha}$ has odd order for every $g \in G$. Then $[G, \alpha] \leq O(G)$.

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Some conditions on solubility for $[G, \alpha]$

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Theorem 3

Let G be a finite group admitting a coprime automorphism α. If any pair of elements from $I_G(\alpha)$ generates a soluble subgroup, then $[G, \alpha]$ is soluble.

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Theorem 4

Let G be a finite group admitting a coprime automorphism α. Then $[G, \alpha]$ is soluble if and only if any subgroup generated by a pair of elements of coprime orders from $J_G(\alpha)$ is soluble.

Observation: If $\langle \beta \rangle$ is any nontrivial subgroup of $\langle \alpha \rangle$, then the collection of β-invariant Sylow subgroups could be larger than the collection of α-invariant Sylow subgroups of G.

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To overcome this issue we will work in the following setting:

Let G be a finite group admitting a coprime group of automorphisms A, let $\alpha \in A$ and let $J_{G,A}(\alpha)$ denote the set of all commutators $[x, \alpha]$, for x in an A-invariant Sylow subgroup of G.

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Let G be a finite group admitting a coprime group of automorphisms A and let $\alpha \in A$. Then $[G, \alpha]$ is soluble if and only if any subgroup generated by a pair of elements of coprime orders from $J_{G,A}(\alpha)$ is soluble.

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Remark: The result in the above theorem fails without the coprimeness assumption. For instance, take α a transposition in $G = S_n$, for $n \geq 5$. Then any pair of elements from $I_G(\alpha)$ generates a soluble subgroup, while $[G, \alpha]$ is insoluble.

Insight of the proof Thm 5

Our goal: to show that there are A-invariant subgroups P and Q of coprime prime power orders such that $[x, \alpha]$ and $[y, \alpha]$ generate an insoluble subgroup for some $x \in P$ and $y \in Q$.

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Suppose the this is false and let $G = [G, A]$ be a counterexample of minimal order. So, any subgroup generated by a pair of elements of coprime order from $J_{G,A}(\alpha)$ is soluble but $[G, \alpha]$ is insoluble.

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If G contains a non-trivial characteristic soluble subgroup M, then by minimality the result holds for A acting on G/M and $[G, A]M/M$ is soluble. Hence $[G, A]$ is soluble too, a contradiction. We may assume that G contains no soluble non-trivial normal subgroups.

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Then $F^*(G)$ is a direct product of non-abelian simple groups, and A acts faithfully on $F^*(G)$ (since $F^*(GA) = F^*(G)$). Choose a component S of $F^*(G)$ such that α does not centralize S. Let $T = S_1 \times \cdots \times S_t$ be the A-orbit of $S = S_1$. Then $[T, \alpha]$ is insoluble and we can assume that $G = T$.

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$G = S_1 \times \cdots \times S_t$, $S = S_1$ and let $A_1 = N_A(S)$. Observe that $A_1/C_A(S)$ is a coprime group of automorphisms of S.

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If $A_1 = 1$, then $|A| = t$ and A acts on G by permuting the coordinates of $S_1 \times \cdots \times S_t$

Theorem (Guralnick-Tiep, 2015)

Let G be a finite group. Then G is soluble if and only if $x_1 x_2 x_3 \neq 1$ for all nontrivial p_i-elements x_i of G for distinct primes p_i, $i = 1, 2, 3$

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So there exist distinct primes p and q, a Sylow p-subgr. P_1 and a Sylow q-subgr. Q_1 of S_1, $x_1 \in P_1$ and $y_1 \in Q_1$ such that $\langle x_1, y_1 \rangle$ is insoluble.

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the direct product Q of the distinct A-conjugates of Q_1, is an A-invariant Sylow q-subgr. of G.
Note that $[x_1, \alpha] \in [P, \alpha]$, $[y_1, \alpha] \in [Q, \alpha]$ and $\langle [x_1, \alpha], [y_1, \alpha] \rangle$ is insoluble.

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the centralizer $C_S(A_1) = L(q_0)$ is a group of the same Lie type defined over the subfield of $q_0 = p^{s/e}$ elements, if e is the order of A_1.

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After some work we are reduced to consider the cases:

- $S = \mathrm{PSL}_2(q)$ with $q = p^s$ for s odd and $s \geq 5$ or
- $S = \mathrm{Sz}(q)$ with $q = 2^s$ for odd $s > 1$.

Primitive prime divisors

For $q = p^s$, a power of a prime p and for any positive integer n, recall that a prime r is said to be a primitive prime divisor of $q^n - 1$ if r divides $q^n - 1$ and r does not divide $q^k - 1$ for any positive integer $k < n$.

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Primitive prime divisors of $q^n + 1$ are defined in a similar way.

The following result is on the existence of primitive prime divisors.

Theorem (Zsigmondy, 1892)

Let $a > b > 0$, $\gcd(a, b) = 1$ and $n > 1$ be positive integers. Then

(i) $a^n - b^n$ has a prime divisor that does not divide $a^k - b^k$ for all positive integers $k < n$, unless $a = 2, b = 1$ and $n = 6$; or $a + b$ is a power of 2 and $n = 2$.

(ii) $a^n + b^n$ has a prime divisor that does not divide $a^k + b^k$ for all positive integers $k < n$, with exception $2^3 + 1^3$.

Let $S = \mathrm{PSL}_2(q)$ with $q = p^s$ for s odd and $s \geq 5$.
Recall that $G = S_1 \times \cdots \times S_t$, $S = S_1$ and $A_1 = N_A(S) \neq 1$.

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Criteria for nilpotency of $[G, \alpha]$

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Theorem 6

Let G be a finite group admitting a coprime group of automorphisms A, and let $\alpha \in A$. Then the following statements are equivalent.

(i) The subgroup $[G, \alpha]$ is nilpotent;

(ii) Any subgroup generated by a pair of elements of coprime orders from $J_{G,A}(\alpha)$ is nilpotent;

(iii) Any subgroup generated by a pair of elements of coprime orders from $J_{G,A}(\alpha)$ is abelian;

(iv) If x and y are elements of coprime orders from $J_{G,A}(\alpha)$, then $|xy| = |x||y|$;

(v) If x and y are elements of coprime orders from $J_{G,A}(\alpha)$, then $\pi(xy) = \pi(x) \cup \pi(y)$.

where $\pi(x)$ denotes the set of prime divisors of the order $|x|$ of x in G.

A consequence that was a motivation

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Form Theorem 6 we can deduce the following variation of the Baumslag-Wiegold result.

Theorem 7

Let G be a finite group admitting a coprime automorphism α. Then $[G, \alpha]$ is nilpotent if, and only if, $|xy| = |x||y|$ whenever x and y are elements of coprime prime power orders from $I_G(\alpha)$.

Recall that $I_G(\alpha)$ is the set of all commutators $g^{-1}g^{\alpha}$, where $g \in G$.

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Remark: at the beginning it was unclear whether the hypothesis on the orders of elements in Theorem 7 is inherited by quotient groups. To overcome this issue we started working with elements of coprime orders from $J_G(\alpha)$ and $J_{G,A}(\alpha)$.

More details can be found in

- C.Acciarri,R.M.Guralnick,and P.Shumyatsky,Coprime automorphisms of finite groups, Trans. Amer. Math. Soc. (2022). 375:7, 4549–4565. https://doi.org/10.1090/tran/8553
- C.Acciarri,R.M.Guralnick,and P.Shumyatsky, Criteria for solubility and nilpotency of finite groups with automorphisms, Bull. London Math. Soc. 2023; 55:3, 1340–1346. https://doi.org/10.1112/blms.12794

Thank you!

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