The McKay conjecture and its Galois refinement

Carolina Vallejo Rodríguez

Università degli Studi di Firenze

Advances in Group theory and Applications, Lecce 2023

Basic Setting

- G finite group.
- $\chi \in \mathrm{Irr}(G)$: **irreducible** complex **characters** of G.

Basic Setting

- G finite group.
- $\chi \in \mathrm{Irr}(G)$: **irreducible** complex **characters** of G.

Character

$\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ homomorphism such that $\chi\colon G \mapsto \mathbb{C}$; $\chi(g) = Tr(\rho(g))$. The **degree** of χ is $\chi(1) = n$.

Basic Setting

- G finite group.
- $\chi \in \mathrm{Irr}(G)$: **irreducible** complex **characters** of G.

Character

$\rho \colon G \to \mathrm{GL}_n(\mathbb{C})$ homomorphism such that $\chi \colon G \mapsto \mathbb{C}; \chi(g) = Tr(\rho(g))$. The **degree** of χ is $\chi(1) = n$.

Irreducible

χ is not the sum of two characters.

Basic Setting

- G finite group.
- $\chi \in \mathrm{Irr}(G)$: **irreducible** complex **characters** of G.

Character

$\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ homomorphism such that $\chi\colon G \mapsto \mathbb{C}; \chi(g) = Tr(\rho(g))$. The **degree** of χ is $\chi(1) = n$.

Irreducible

χ is not the sum of two characters.

Examples:

- $\mathbf{1}_G(g) = 1$ is the trivial character of G.

Basic Setting

- G finite group.
- $\chi \in \mathrm{Irr}(G)$: **irreducible** complex **characters** of G.

Character

$\rho\colon G \to \mathrm{GL}_n(\mathbb{C})$ homomorphism such that $\chi\colon G \mapsto \mathbb{C}$; $\chi(g) = Tr(\rho(g))$. The **degree** of χ is $\chi(1) = n$.

Irreducible

χ is not the sum of two characters.

Examples:

- $\mathbf{1}_G(g) = 1$ is the trivial character of G.
- $\mathrm{Lin}(G) = \mathrm{Hom}(G, \mathbb{C}^*) \subseteq \mathrm{Irr}(G)$. Note $|\mathrm{Lin}(G)| = |G/G'|$, here $G' = [G, G]$.

Character Theory as a tool in Group Theory

Character Theory as a tool in Group Theory

- Burnside's $p^a q^b$ Theorem: Every group of order divisible by at most two primes is solvable.

Character Theory as a tool in Group Theory

- Burnside's $p^a q^b$ Theorem: Every group of order divisible by at most two primes is solvable.
- Feit-Thompson's Odd-Order Theorem: Every group of odd degree is solvable.

Character Theory as a tool in Group Theory

- Burnside's $p^a q^b$ Theorem: Every group of order divisible by at most two primes is solvable.
- Feit-Thompson's Odd-Order Theorem: Every group of odd degree is solvable.
- The Classification of Finite Simple Groups (CFSG): Every finite simple groups falls in one of the following families:
  - Groups of prime order.
  - Alternating groups.
  - Groups of Lie type.
  - 26 sporadic groups.

The McKay Conjecture

The McKay Conjecture

We fix a prime p and write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \gcd(\chi(1), p) = 1\} \supseteq \mathrm{Lin}(G)$.

The McKay Conjecture

We fix a prime p and write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \gcd(\chi(1), p) = 1\} \supseteq \mathrm{Lin}(G)$.

John McKay observed that for $G = \mathsf{S}_n$ and $p = 2$

$$|\mathrm{Irr}_{2'}(G)|$$

is always a power of 2.

The McKay Conjecture

We fix a prime p and write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \gcd(\chi(1), p) = 1\} \supseteq \mathrm{Lin}(G)$.

John McKay observed that for $G = \mathrm{S}_n$ and $p = 2$

$$|\mathrm{Irr}_{2'}(G)|$$

is always a power of 2.

The McKay Conjecture, 1972

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. There is a a bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)) .$$

The McKay Conjecture

We fix a prime p and write $\mathrm{Irr}_{p'}(G) = \{\chi \in \mathrm{Irr}(G) \mid \gcd(\chi(1), p) = 1\} \supseteq \mathrm{Lin}(G)$.

John McKay observed that for $G = \mathrm{S}_n$ and $p = 2$

$$|\mathrm{Irr}_{2'}(G)|$$

is always a power of 2.

The McKay Conjecture, 1972

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. There is a a bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)) .$$

Note that $P \in \mathrm{Syl}_2(S_n)$ is self-normalizing so

$$|\mathrm{Irr}_{2'}(\mathrm{S}_n)| = |\mathrm{Irr}_{2'}(P)| = |\mathrm{Lin}(P)| = |P/P'| .$$

The McKay Conjecture: State of the art

1970-2000: The McKay Conjecture holds for p-solvable groups, symmetric, alternating and general linear groups, sporadic groups, etc.

The McKay Conjecture: State of the art

1970-2000: The McKay Conjecture holds for p-solvable groups, symmetric, alternating and general linear groups, sporadic groups, etc.

What about using the CFSG?

The McKay Conjecture: State of the art

1970-2000: The McKay Conjecture holds for p-solvable groups, symmetric, alternating and general linear groups, sporadic groups, etc.

What about using the CFSG?

Theorem (Isaacs-Malle-Navarro, 2007)

Let G be a finite group and p a prime. If every simple non-abelian group **involved** in G satisfies the **inductive McKay conditions** for p, then G satisfies the statement of the McKay conjecture for p.

The McKay Conjecture: Reduction Theorem and Inductive Conditions

The Isaacs-Malle-Navarro Reduction Theorem: In order to prove the McKay conjecture for p it is enough to check that all finite simple groups satisfy the inductive McKay conditions for p

The McKay Conjecture: Reduction Theorem and Inductive Conditions

The Isaacs-Malle-Navarro Reduction Theorem: In order to prove the McKay conjecture for p it is enough to check that all finite simple groups satisfy the inductive McKay conditions for p $\longrightarrow$ Use the CFSG!

The McKay Conjecture: Reduction Theorem and Inductive Conditions

The Isaacs-Malle-Navarro Reduction Theorem: In order to prove the McKay conjecture for p it is enough to check that all finite simple groups satisfy the inductive McKay conditions for p $\longrightarrow$ Use the CFSG!

The inductive McKay conditions for S and p: Too technical! A stronger form of the conjecture holds for the universal covering group X of S.

The McKay Conjecture: Reduction Theorem and Inductive Conditions

The Isaacs-Malle-Navarro Reduction Theorem: In order to prove the McKay conjecture for p it is enough to check that all finite simple groups satisfy the inductive McKay conditions for p $\longrightarrow$ Use the CFSG!

The inductive McKay conditions for S and p: Too technical! A stronger form of the conjecture holds for the universal covering group X of S.

Theorem (Malle-Spaeth, 2016)

The McKay conjecture holds for $p = 2$.

The McKay Conjecture: Reduction Theorem and Inductive Conditions

The Isaacs-Malle-Navarro Reduction Theorem: In order to prove the McKay conjecture for p it is enough to check that all finite simple groups satisfy the inductive McKay conditions for p $\longrightarrow$ Use the CFSG!

The inductive McKay conditions for S and p: Too technical! A stronger form of the conjecture holds for the universal covering group X of S.

Theorem (Malle-Spaeth, 2016)

The McKay conjecture holds for $p = 2$.

Last year, the following was announced during Gunter Malle's 60 birthday conference.

Theorem (Spaeth)

The McKay conjecture holds for $p = 3$...

More on the McKay conjecture

What more can we say about a "McKay bijection" $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

More on the McKay conjecture

What more can we say about a "McKay bijection" $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

$\chi \in \mathrm{Irr}_{p'}(G)$ consider the restriction to subgroup $M \leqslant G$

$$\psi = \chi|_M .$$

ψ is a sum of irreducible characters of M, called **constituents**

More on the McKay conjecture

What more can we say about a "McKay bijection" $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

$\chi \in \mathrm{Irr}_{p'}(G)$ consider the restriction to subgroup $M \leqslant G$

$$\psi = \chi|_M .$$

ψ is a sum of irreducible characters of M, called **constituents** $\to$ some constituent has degree coprime to p.

More on the McKay conjecture

What more can we say about a "McKay bijection" $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

$\chi \in \mathrm{Irr}_{p'}(G)$ consider the restriction to subgroup $M \leqslant G$

$$\psi = \chi|_M .$$

ψ is a sum of irreducible characters of M, called **constituents** $\to$ some constituent has degree coprime to p.

Suppose that $\mathbf{N}_G(P) = P$. Then

$$\mathrm{Irr}_{p'}(P) = \mathrm{Lin}(P) = \{\lambda \in \mathrm{Irr}(P) \mid \lambda(1) = 1\} .$$

Special McKay bijections: the self-normalizing case

Theorem (Navarro-Tiep-V., 2014)

Let G be a finite group, p an odd prime and $P \in \mathrm{Syl_p(G)}$. Suppose that $\mathbf{N}_G(P) = P$. If $\chi \in \mathrm{Irr}_{p'}(G)$, then

$$\chi|_P = \chi^* + \Delta,$$

where χ^* has degree 1 and the constituents of Δ have degree divisible by p. Moreover

$$\chi \mapsto \chi^*$$

defines a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Lin}(P)$.

Special McKay bijections: the self-normalizing case

Theorem (Navarro-Tiep-V., 2014)

Let G be a finite group, p an odd prime and $P \in \mathrm{Syl_p(G)}$. Suppose that $\mathbf{N}_G(P) = P$. If $\chi \in \mathrm{Irr}_{p'}(G)$, then

$$\chi|_P = \chi^* + \Delta,$$

where χ^* has degree 1 and the constituents of Δ have degree divisible by p. Moreover

$$\chi \mapsto \chi^*$$

defines a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Lin}(P)$.

Corollary

The McKay conjecture holds at p for groups with a self-normalizing Sylow p-subgroup.

Special McKay bijections: the self-normalizing case

The statement of NTV'14 does not hold for $p = 2$. (Recall that symmetric groups S_n have self-normalizing Sylow 2-subgroups.)

Special McKay bijections: the self-normalizing case

The statement of NTV'14 does not hold for $p = 2$. (Recall that symmetric groups S_n have self-normalizing Sylow 2-subgroups.)

Example: If $\chi \in \mathrm{Irr}(S_5)$ corresponds to the partition $(3,2)$ then

$$\chi|_P = \mathbf{1}_P + \lambda + \mu + \Delta\,,$$

$\lambda, \mu \in \mathrm{Lin}(P)$ and $\Delta \in \mathrm{Irr}(P)$ has degree 2.

Special McKay bijections: the self-normalizing case

The statement of NTV'14 does not hold for $p = 2$. (Recall that symmetric groups S_n have self-normalizing Sylow 2-subgroups.)

Example: If $\chi \in \mathrm{Irr}(S_5)$ corresponds to the partition $(3, 2)$ then

$$\chi|_P = \mathbf{1}_P + \lambda + \mu + \Delta\,,$$

$\lambda, \mu \in \mathrm{Lin}(P)$ and $\Delta \in \mathrm{Irr}(P)$ has degree 2.

For symmetric groups and $p = 2$, Giannelli and Giannelli-Kleshchev-Navarro-Tiep constructed McKay bijections compatible with character restriction.

Character Values: The Galois refinement of the McKay conjecture

Are there further relations between character values in the context of the McKay conjecture?

$$\mathrm{Irr}_{p'}(G) \xrightarrow{1:1} \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

Character Values: The Galois refinement of the McKay conjecture

Are there further relations between character values in the context of the McKay conjecture?

$$\mathrm{Irr}_{p'}(G) \xrightarrow{1:1} \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

Note that $\chi(g) \in \mathbb{Q}(e^{2\pi i/n})$ where $n = |G|$. Moreover, if $\sigma \in \Gamma := \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ then $\chi^\sigma \in \mathrm{Irr}(G)$ where $\chi^\sigma(g) = \sigma(\chi(g))$.

Character Values: The Galois refinement of the McKay conjecture

Are there further relations between character values in the context of the McKay conjecture?

$$\mathrm{Irr}_{p'}(G) \xrightarrow{1:1} \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

Note that $\chi(g) \in \mathbb{Q}(e^{2\pi i/n})$ where $n = |G|$. Moreover, if $\sigma \in \Gamma := \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ then $\chi^\sigma \in \mathrm{Irr}(G)$ where $\chi^\sigma(g) = \sigma(\chi(g))$.

The group Γ acts on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$

Character Values: The Galois refinement of the McKay conjecture

Are there further relations between character values in the context of the McKay conjecture?

$$\mathrm{Irr}_{p'}(G) \xrightarrow{1:1} \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$$

Note that $\chi(g) \in \mathbb{Q}(e^{2\pi i/n})$ where $n = |G|$. Moreover, if $\sigma \in \Gamma := \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$ then $\chi^\sigma \in \mathrm{Irr}(G)$ where $\chi^\sigma(g) = \sigma(\chi(g))$.

The group Γ acts on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$

Are these actions permutation isomorphic?

Can we always find Γ-equivariant McKay bijections $\mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

Can we always find Γ-equivariant McKay bijections $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

• The bijection $\chi \mapsto \chi^*$ given by NTV'14 in the self-normalizing case is Γ-equivariant! Just notice that $(\chi^\sigma)_P = (\chi|_P)^\sigma = (\chi^*)^\sigma + \Delta^\sigma$ so $(\chi^\sigma)^* = (\chi^*)^\sigma$ for every $\sigma \in \Gamma$.

Can we always find Γ-equivariant McKay bijections $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

• The bijection $\chi \mapsto \chi^*$ given by NTV'14 in the self-normalizing case is Γ-equivariant! Just notice that $(\chi^\sigma)_P = (\chi|_P)^\sigma = (\chi^*)^\sigma + \Delta^\sigma$ so $(\chi^\sigma)^* = (\chi^*)^\sigma$ for every $\sigma \in \Gamma$. (This means $\chi(g) \in \mathbb{Z}$ all $g \in G$ of order coprime to p!!)

Can we always find Γ-equivariant McKay bijections $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

• The bijection $\chi \mapsto \chi^*$ given by NTV'14 in the self-normalizing case is Γ-equivariant! Just notice that $(\chi^\sigma)_P = (\chi|_P)^\sigma = (\chi^*)^\sigma + \Delta^\sigma$ so $(\chi^\sigma)^* = (\chi^*)^\sigma$ for every $\sigma \in \Gamma$. (This means $\chi(g) \in \mathbb{Z}$ all $g \in G$ of order coprime to p!!)

• The answer is NO in general.

Can we always find Γ-equivariant McKay bijections $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$?

• The bijection $\chi \mapsto \chi^*$ given by NTV'14 in the self-normalizing case is Γ-equivariant! Just notice that $(\chi^\sigma)_P = (\chi|_P)^\sigma = (\chi^*)^\sigma + \Delta^\sigma$ so $(\chi^\sigma)^* = (\chi^*)^\sigma$ for every $\sigma \in \Gamma$. (This means $\chi(g) \in \mathbb{Z}$ all $g \in G$ of order coprime to p!!)

• The answer is NO in general. Consider $G = \mathsf{S}_5$ and $p = 5$. Then $\mathbf{N}_G(P) = \mathsf{C}_5 \rtimes \mathsf{C}_4$. The 5 irreducible characters of degree coprime to 5 of G are rational, while $\mathsf{C}_5 \rtimes \mathsf{C}_4$ has 2 irreducible characters of degree 1 whose different values are the 4th roots of unity $\{\pm 1, \pm i\}$.

The Galois refinement: The group $\mathcal{H}_p$

Let $n = |G|$ and write $\Gamma = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$.

The Galois refinement: The group $\mathcal{H}_p$

Let $n = |G|$ and write $\Gamma = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$.

Definition (Navarro, 2004)

Let p be a prime. Define $\mathcal{H}_p \leqslant \Gamma$ as the subgroup consisting of those Galois automorphisms $\sigma \in \Gamma$ for which there is a fixed $f \in \mathbb{N}$ such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity in $\mathbb{Q}(e^{2\pi i/n})$ of order coprime to p.

The Galois refinement: The group $\mathcal{H}_p$

Let $n = |G|$ and write $\Gamma = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$.

Definition (Navarro, 2004)

Let p be a prime. Define $\mathcal{H}_p \leqslant \Gamma$ as the subgroup consisting of those Galois automorphisms $\sigma \in \Gamma$ for which there is a fixed $f \in \mathbb{N}$ such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity in $\mathbb{Q}(e^{2\pi i/n})$ of order coprime to p.

- $\mathcal{H}_p$ is the stabilizer in Γ of any prime ideal over p in $\mathbb{Z}[e^{2\pi i/n}]$.

The Galois refinement: The group $\mathcal{H}_p$

Let $n = |G|$ and write $\Gamma = \mathrm{Gal}(\mathbb{Q}(e^{2\pi i/n})/\mathbb{Q})$.

Definition (Navarro, 2004)

Let p be a prime. Define $\mathcal{H}_p \leqslant \Gamma$ as the subgroup consisting of those Galois automorphisms $\sigma \in \Gamma$ for which there is a fixed $f \in \mathbb{N}$ such that $\sigma(\xi) = \xi^{p^f}$ for every root of unity in $\mathbb{Q}(e^{2\pi i/n})$ of order coprime to p.

- $\mathcal{H}_p$ is the stabilizer in Γ of any prime ideal over p in $\mathbb{Z}[e^{2\pi i/n}]$.
- $\mathcal{H}_p$ is the Galois group of the n-th cyclotomic extension of the p-adic numbers.

The Galois refinement: The McKay-Navarro Conjecture

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl_p(G)}$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))\,.$$

The Galois refinement: The McKay-Navarro Conjecture

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl_p(G)}$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))\,.$$

The above is equivalent to saying that the actions of $\mathcal{H}_p$ on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ are permutation isomorphic.

The Galois refinement: The McKay-Navarro Conjecture

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl_p(G)}$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)) \,.$$

The above is equivalent to saying that the actions of $\mathcal{H}_p$ on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ are permutation isomorphic.

Example: if $\sigma \in \mathcal{H}_5$ then $\sigma(i) = i^{5^f} = i$, so every McKay bijection for S_5 and $p = 5$ is $\mathcal{H}_5$-equivariant (in this case the action of $\mathcal{H}_5$ on both "sides" is trivial).

The Galois refinement: The McKay-Navarro Conjecture

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl_p(G)}$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)) .$$

The above is equivalent to saying that the actions of $\mathcal{H}_p$ on $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ are permutation isomorphic.

Example: if $\sigma \in \mathcal{H}_5$ then $\sigma(i) = i^{5^f} = i$, so every McKay bijection for S_5 and $p = 5$ is $\mathcal{H}_5$-equivariant (in this case the action of $\mathcal{H}_5$ on both "sides" is trivial).

Notice that the McKay-Navarro Conjecture implies the McKay Conjecture.

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)) \,.$$

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P))\,.$$

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)).$$

The McKay-Navarro conjecture has disclosed new unexpected connections between group and character theory: e.g. its statement implies that the exponent of P/P' can be detected in the character table (open for odd primes).

The McKay-Navarro Conjecture (Navarro, 2004)

Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. There exists a $\mathcal{H}_p$-equivariant bijection

$$\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathbf{N}_G(P)) \, .$$

The McKay-Navarro conjecture has disclosed new unexpected connections between group and character theory: e.g. its statement implies that the exponent of P/P' can be detected in the character table (open for odd primes).

- Sporadic groups, symmetric groups (Navarro, 2004) ✓
- p-solvable groups (Turull, 2013) ✓
- Alternating groups (Brunat-Nath, 2021) ✓
- Simple groups of Lie type in defining characteristic (Ruhstorfer, 2021) ✓

The Galois refinement: A Reduction Theorem

The Galois refinement: A Reduction Theorem

Theorem (Navarro-Spaeth-V., 2020)

Let G be a finite group and p a prime. If every simple non-abelian group involved in G satisfies the **inductive McKay-Navarro conditions** for p, then the group G satisfies the McKay-Navarro Conjecture for p.

The Galois refinement: A Reduction Theorem

Theorem (Navarro-Spaeth-V., 2020)

Let G be a finite group and p a prime. If every simple non-abelian group involved in G satisfies the **inductive McKay-Navarro conditions** for p, then the group G satisfies the McKay-Navarro Conjecture for p.

In particular, in order to prove the McKay-Navarro conjecture for p in full generality, it is enough to check a set of conditions with respect to p for the finite simple non-abelian groups.

The Galois refinement: A Reduction Theorem

Theorem (Navarro-Spaeth-V., 2020)

Let G be a finite group and p a prime. If every simple non-abelian group involved in G satisfies the **inductive McKay-Navarro conditions** for p, then the group G satisfies the McKay-Navarro Conjecture for p.

In particular, in order to prove the McKay-Navarro conjecture for p in full generality, it is enough to check a set of conditions with respect to p for the finite simple non-abelian groups.

The inductive McKay-Navarro conditions are harder to prove than the inductive McKay conditions.

The Galois refinement: A Reduction Theorem

Theorem (Navarro-Spaeth-V., 2020)

Let G be a finite group and p a prime. If every simple non-abelian group involved in G satisfies the **inductive McKay-Navarro conditions** for p, then the group G satisfies the McKay-Navarro Conjecture for p.

In particular, in order to prove the McKay-Navarro conjecture for p in full generality, it is enough to check a set of conditions with respect to p for the finite simple non-abelian groups.

The inductive McKay-Navarro conditions are harder to prove than the inductive McKay conditions. The following was announced in Oberwolfach last April.

Theorem (Ruhstorfer-Schaeffer Fry)

The McKay-Navarro Conjecture holds for $p = 2$.

Thanks for your attention!