Subnormality in Linear Groups

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J. Pure Appl. Algebra (2023), no. 2, Paper No. 107185.

Permutable subgroups in linear groups

to appear

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Groups whose proper subgroups are linear
J. Algebra **592** (2022), 153–168.

The upper and lower central series in linear groups
Q.J. Math. **73** (2022), 261–275.

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Motivation and History

Normality is **not** a **transitive relation**

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A **subgroup** X of a **group** G is **subnormal** in G if there is a finite **chain** of subgroups

$$X = X_0 \trianglelefteq X_1 \trianglelefteq \ldots \trianglelefteq X_n = G$$

connecting X to G.

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A **subgroup** X of a (**finite**) **group** G is **subnormal** in G
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> "**Subnormal subgroups** are the *bare bones* or *skeleton* of a group, providing the framework for all other structures"
>
> **Philip Hall**

Helmut Wielandt (1939)

Eine Verallgemeinerung der invarianten Untergruppen
Math. Z. **45** (1939), no. 1, 209–244.

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Join Theorem

The subgroup generated by two subnormal subgroups of a **finite** group is itself subnormal.

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Join Theorem

The subgroup generated by two subnormal subgroups of a **finite** group is itself subnormal.

Hans Zassenahus (1958)

There exists a group having two subnormal subgroups whose join is not subnormal.

Let G be a **group** and let $X \leqslant G$.

- X is **ascendant** if there is an ascending series of subgroups

$$X = X_0 \trianglelefteq X_1 \trianglelefteq \ldots X_\alpha \trianglelefteq X_{\alpha+1} \trianglelefteq \ldots X_\lambda = G$$

- X is **descendant** if there is a descending series of subgroups

$$X = X_\lambda \ldots \trianglelefteq X_{\alpha+1} \trianglelefteq X_\alpha \ \ldots \trianglelefteq X_1 \trianglelefteq X_0 = G$$

- X is **serial** if there is a chain of subgroups between X and G such that if H and K are consecutive subgroups, then $H \trianglelefteq K$.

Let G be a **group**. A subgroup X of G is said to be **serial** in G if there is a set

$$\{(\Lambda_\sigma, V_\sigma) \; : \; \sigma \in \Sigma\}$$

of subgroups of G such that

(i) Σ is a totally ordered indexing set;

(ii) $H \leqslant V_\sigma \trianglelefteq \Lambda_\sigma, \forall \sigma \in \Sigma$;

(iii) $\Lambda_\sigma \leqslant V_\tau$ if $\sigma < \tau$;

(iv) $G \setminus H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$.

A subgroup X is **ascendant** (resp. **descendant**) if it is serial and Σ is well-ordered (resp. inversely well-ordered).

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Every finite *or* nilpotent *subgroup of a* **hypocentral** *group is* **descendant**.

- In particular, every **cyclic** subgroup of a hypocentral group is **descendant**.
- Note that every **free group** is **hypocentral**.

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Hickin & Phillips (1973)

A join of arbitrarily many subnormal subgroups is serial.

What about linear groups?

Free groups *are linear.*

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- ↓ There is $N \triangleleft F$ such that F/N is isomorphic to the **Zassenhaus group**
- ↓ Let H/N and K/N be **subnormal subgroups** of F/N whose join is **not** subnormal
- □ Then H and K are **subnormal subgroups** of F whose join is **not** subnormal

> *Let* G *be a* **linear group** *all of whose finitely generated subgroups are* **serial**. *Then* G *is* **hypercentral**.

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□ Thus, G is **hypercentral**.

Let G *be a* **linear group** *all of whose finitely generated subgroups are* **serial**. *Then* G *is* **hypercentral**.

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- G is **hypercentral**
- ↓ There is a positive integer k such that $G/Z_k(G)$ is **periodic**
- ↓ *A periodic linear group whose cyclic subgroups are descendant must be nilpotent*
- □ Thus, $G/Z_k(G)$ is nilpotent and so G is **nilpotent**.

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- We can bound the **ascendancy length** in terms of the degree of G as a linear group
- Proof splits in several cases and is accomplished in a series of lemmas such as: *If* N *is unipotent and normal, then* X *is subnormal in* XN

Let G be a **periodic linear group** of **degree** 2 over a field of **characteristic** 2. Then every serial subgroup is subnormal of defect at most $2 + \left[\log_2\left(\theta(2)\right)\right] + \left[\log_2\left(\mu(2)\right)\right]$.

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- If either the characteristic or the degree are > 2, then this is not true.

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Let G be a **periodic linear group**. Then the join of arbitrarily many **subnormal** subgroups of G is **subnormal**.

Let G be a **soluble-by-periodic linear group**. Then the join of finitely many **subnormal** subgroups of G is **subnormal**.

A subgroup X of a group G is **f-subnormal** if there is a finite chain of subgroups

$$X = X_0 \leqslant X_1 \leqslant \ldots \leqslant X_n = G$$

such that $X_i \trianglelefteq X_{i+1}$ or $|X_{i+1} : X_i| < \infty$ for every $0 \leqslant i < n$.

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- Every subgroup of a **finite group** is f-subnormal.

A subgroup X of a group G is **$\mathfrak{f}$-subnormal** if there is a finite chain of subgroups

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- Every subgroup of a **finite group** is $\mathfrak{f}$-subnormal.
- Every subnormal subgroup is $\mathfrak{f}$-subnormal.

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Let G be a group and let X be a subgroup of G. Then X is **subnormal** if and only if it is **ascendant** and **$\mathfrak{f}$-subnormal**.

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Let G be a group and let X be a subgroup of G. Then X is **subnormal** if and only if it is **ascendant** and **$\mathfrak{f}$-subnormal**.

The join of two **$\mathfrak{f}$-subnormal** subgroups may **fail** to be $\mathfrak{f}$-subnormal even in periodic linear groups.

Let σ be a partition of $\mathbb{P}$. A subgroup X of a group G is **σ-subnormal** if there is a finite chain of subgroups

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such that $X_{i+1}/(X_i)_{X_{i+1}}$ is a σ_i-group for some $\sigma_i \in \sigma$ and for every $0 \leqslant i < n$.

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The join of arbitrarily many **σ-subnormal** subgroups is **σ-subnormal** in periodic linear groups.

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The join of arbitrarily many **σ-subnormal** subgroups is **σ-subnormal** in periodic linear groups.

- **σ-subnormality** coincides with **σ-seriality** in periodic linear groups.

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M. Ferrara • M.T.

σ-Subnormality in locally finite groups

J. Algebra 614 (2023), 867–897.

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Let G be a **periodic linear** group. Then every **permutable** subgroup of G is **subnormal**

Every **permutable** subgroup of a group is **ascendant**.

Let G be a **soluble-by-periodic linear** group over the field F. If either $\mathrm{char}(F) \neq 0$ or $u(G) = \{1\}$, then every **permutable** subgroup of G is **subnormal**

. . . Fin

. . . Thank You . . .