Rota-Baxter operators on Clifford semigroups and the Yang-Baxter equation

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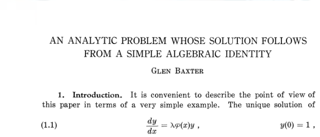
The Baxters are not the same...

Glen Earl Baxter (1930-1983) was an American mathematician. His research fields include probability theory and combinatorial analysis.



AN ANALYTIC PROBLEM WHOSE SOLUTION FOLLOWS FROM A SIMPLE ALGEBRAIC IDENTITY

GLEN BAXTER

1. Introduction. It is convenient to describe the point of view of this paper in terms of a very simple example. The unique solution of

$$\frac{dy}{dx} = \lambda\varphi(x)y\,, \qquad y(0) = 1\,, \tag{1.1}$$

Figure: Gian-Carlo Rota and G.E. Baxter

Rodney James Baxter (born in 1940) is an Australian physicist, specializing in statistical mechanics.

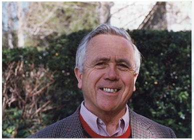


Figure: Chen-Ning Yang and R.J. Baxter

Rota-Baxter operators and applications

Rota–Baxter operators on commutative algebras first appeared in 1960 in Baxter's probability studies and were subsequently investigated by several authors, including Rota:

- G. Baxter, *An analytic problem whose solution follows from a simple algebraic identity*, Pac. J. Math. 10 (1960) 731–742.
- G.-C. Rota, *Baxter algebras and combinatorial identities*, I, II, Bull. Am. Math. Soc. 75 (1969) 325–329, Bull. Am. Math. Soc. 75 (1969) 330–334

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They have connections with Mathematical Physics, number theory, Hopf algebras, combinatorics, et cetera, as one can see in:

- L. Guo, *An Introduction to Rota-Baxter Algebra*, Surveys of Modern Mathematics, vol. 4, International Press/Higher Education Press, Somerville (MA, USA)/Beijing, 2012

RB-operators on groups

The notion of Rota-Baxter on groups was studied in two papers:

L. Guo, H. Lang, Y. Sheng, *Integration and geometrization of Rota-Baxter Lie algebras*, Adv. Math. 387 (2021), Paper No. 107834, 34 pp.

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Definition

Let $(G, +)$ be a group. A map $\mathfrak{R} : G \to G$ is a *Rota–Baxter operator* if

$$\forall a, b \in G \qquad \mathfrak{R}(a) + \mathfrak{R}(b) = \mathfrak{R}(a + \mathfrak{R}(a) + b - \mathfrak{R}(a)).$$

RB-operators on groups

Examples

1. If $(G, +)$ is a group, $\mathfrak{E}(a) = 0$ and $\mathfrak{O}(a) = -a$, for every $a \in G$.

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Proposition

Let $(G, +)$ be a group and $\mathfrak{R} : G \to G$ an RB-operator on G.
Then, the following are RB-operators on G:

1. $\tilde{\mathfrak{R}}(a) = -a + \mathfrak{R}(a)$;
2. if $\varphi \in \mathrm{Aut}(G)$, the map $\mathfrak{R}^{(\varphi)} := \varphi^{-1}\mathfrak{R}\varphi$.

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If G is abelian, RB-operators are all the endomorphisms of G.
In general, any endomorphism $\mathfrak{R} : G \to G$ that is an RB-operator on a group G (not necessarily abelian) is called *RB-endomorphism* of G.

Idempotent Rota-Baxter endomorphisms

F. Catino, M. Mazzotta, P. Stefanelli, *Rota-Baxter operators on Clifford semigroups and the Yang-Baxter equation*, J. Algebra 622 (2023) 587–613

Theorem

Let $(G, +)$ be a group and consider:
- $N \trianglelefteq G$ such that G/N is abelian,
- $\mathcal{S}$ a set of representatives of G/N that is a subgroup of G.

Then, any map $\mathfrak{R} : G \to G$ such that

$$\operatorname{Im}\mathfrak{R} = \mathcal{S} \qquad \& \qquad \mathfrak{R}(g) \in N + g,$$

for every $g \in G$, is an idempotent RB-endomorphism of G.
Moreover, every idempotent RB-endomorphism of G is of this form.

Rota-Baxter induced group

Proposition (G.L.S., 2021)

Let $(G,+)$ be a group and $\mathfrak{R}$ an RB-operator on G. Set

$$a \circ_{\mathfrak{R}} b := a + \mathfrak{R}(a) + b - \mathfrak{R}(a),$$

for all $a, b \in G$, then $(G, \circ_{\mathfrak{R}})$ is a group.

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for all $a, b \in G$, then $(G, \circ_{\mathfrak{R}})$ is a group. Moreover,

1. $\mathfrak{R}$ is a RB-operator on $(G, \circ_{\mathfrak{R}})$;
2. the map $\mathfrak{R} : (G, +) \to (G, \circ_{\mathfrak{R}})$ is a homomorphism of groups.

Skew braces coming from RB-operators

V. G. Bardakov, V. Gubarev, *Rota-Baxter groups, skew left braces, and the Yang-Baxter equation*, J. Algebra 596 (2022), 328–351

Proposition (B.G., 2022)

Let $(G, +)$ be a group and $\mathfrak{R}$ an RB-operator on G. Then, $G_{\mathfrak{R}} := (G, +, \circ_{\mathfrak{R}})$ is a skew brace.

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Note that not all skew braces come from RB-operators.

A. Caranti, L. Stefanello, *Skew braces from Rota-Baxter operators: a cohomological characterisation and some examples*, Ann. Mat. Pura Appl. (4) 202 (2023), no. 1, 1–13

Clifford semigroups

Inverse semigroup theory was initiated in the 1950s and it has been extensively studied over the years.

A semigroup S is called *inverse* if, for each $a \in S$, there exists a unique $a^{-1} \in S$ satisfying

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S is a *Clifford semigroup* if it is an inverse semigroup such that, for each $a \in S$,

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RB-operators on Clifford semigroups

Definition (C., M., S., 2023)

If $(S,+)$ is a Clifford semigroup, any map $\mathfrak{R} : S \to S$ satisfying

$$\forall a,b \in S \qquad \begin{aligned} \mathfrak{R}(a) + \mathfrak{R}(b) &= \mathfrak{R}(a + \mathfrak{R}(a) + b - \mathfrak{R}(a)) \\ a + \mathfrak{R}(a) - \mathfrak{R}(a) &= a \end{aligned}$$

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Example

If $\varphi \in \mathrm{End}(S)$ such that $\varphi^2 = \varphi$ and $\varphi(e) = e$, for every $e \in E(S)$, then the map $\mathfrak{R} := -\varphi$ is an RB-operator on S.
As special cases, $\mathfrak{E}(a) = a - a$ and $\mathfrak{O}(a) = -a$, for every $a \in S$.

Rota-Baxter induced Clifford semigroup

Proposition (C., M., S., 2023)

Let $\mathfrak{R}$ an RB-operator on a Clifford semigroup $(S, +)$. Set

$$a \circ_{\mathfrak{R}} b := a + \mathfrak{R}(a) + b - \mathfrak{R}(a),$$

for all $a, b \in S$, then $(S, \circ_{\mathfrak{R}})$ is a Clifford semigroup.

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F. Catino, M. Mazzotta, M.M. Miccoli, P. Stefanelli, *Set-theoretic solutions of the Yang-Baxter equation associated to weak braces*, Semigroup Forum 104 (2) (2022) 228–255

Dual weak braces and the YBE solutions

Definition (C., M., M., S., 2022)

A *dual weak brace* is a triple $(S, +, \circ)$ such that $(S, +)$ and $(S, \circ)$ are Clifford semigroups satisfying

- $\forall a, b, c \in S \quad a \circ (b + c) = a \circ b - a + a \circ c$,
- $\forall a \in S \quad a \circ a^{-} = -a + a$,

where $-a$ and a^{-} denote the inverses of $(S, +)$ and $(S, \circ)$.

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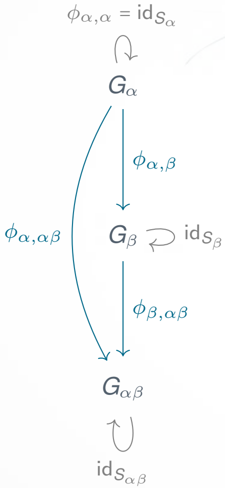
If $(S, +, \circ)$ is a dual weak brace, then the map

$$r_S(a, b) = \left(a \circ (a^{-} + b), (a^{-} + b)^{-} \circ b\right),$$

for all $a, b \in S$, is a set-theoretic solution of the YBE.

Strong semilattice of groups



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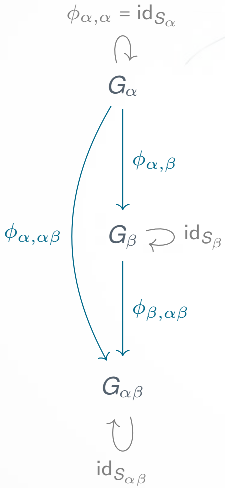

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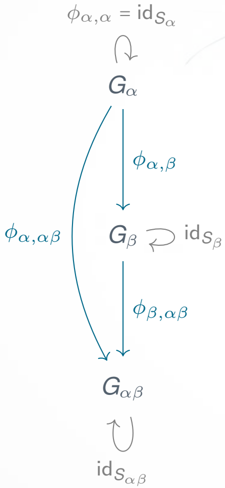

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- Y a (lower) semilattice;
- $\{G_\alpha \mid \alpha \in Y\}$ a family of disjoint groups;
- For each pair α, β of elements of Y such that $\alpha \geq \beta$, let $\phi_{\alpha,\beta} : G_\alpha \to G_\beta$ be a homomorphism of groups such that
  1. $\phi_{\alpha,\alpha}$ is the identical automorphism of G_α, for every $\alpha \in Y$;
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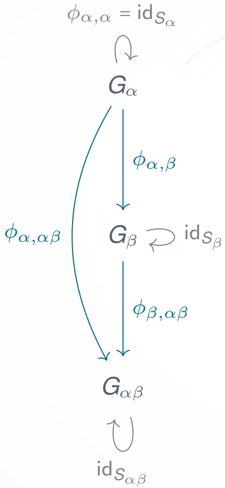

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Then, $S = \bigcup \{G_\alpha \mid \alpha \in Y\}$ endowed with the operation given by

$$\forall a \in G_\alpha, b \in G_\beta \quad ab := \phi_{\alpha,\alpha\beta}(a)\,\phi_{\beta,\alpha\beta}(b)$$

is a Clifford semigroup and every Clifford semigroup can be obtained in this way.

Strong RB-operators

Theorem (C.M.S., 2023)

Let $S = [Y; G_\alpha; \phi\alpha, \beta]$ be a Clifford semigroup and assume that $\mathfrak{R}_\alpha$ is a Rota–Baxter operator on each group $(G_\alpha, +)$, for every $\alpha \in Y$. Then, the map $\mathfrak{R} : S \to S$ given by

$$\mathfrak{R}(a) = \mathfrak{R}_\alpha(a),$$

for every $a \in G_\alpha$, is a RB-operator on $(S, +)$ if and only if the condition

$$\mathfrak{R}_\beta \phi_{\alpha,\beta} = \phi_{\alpha,\beta} \mathfrak{R}_\alpha,$$

is satisfied, for all $\alpha, \beta \in Y$ such that $\alpha \geq \beta$.

Thank you!