f-zpd algebras and a multilinear Nullstellensatz

joint work with Ž. Bajuk, M. Brešar and P. Fagundes

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Algebras

Let F be a fixed field.

Algebra: F-vector space that is also a ring with additional properties relating the 3 operations.

Example

$M_d(F)$ is the algebra of $d \times d$ matrices.

All our algebras are associative and with unit

Zpd-algebras

Definition

An F-algebra A is called zero product determined (zpd) if, for every bilinear functional $\varphi\colon A \times A \to F$ satisfying $\varphi(a,b) = 0$ whenever $ab = 0$, there exists a linear functional $\tau\colon A \to F$ such that

$$\varphi(a,b) = \tau(ab), \ \forall a, b \in A.$$

Example

The algebra $A = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mid \alpha, \beta \in F \right\}$ is not zpd.

Theorem (Brešar, 2021)

$M_d(F)$ is zpd.

Multilinear polynomials

$X = \{x_1, x_2, \ldots\}$ countable set of non-commuting variables.

$F\langle X\rangle$ is the free associative algebra of polynomials on X over F.

Definition

A polynomial $f = f(x_1, \ldots, x_m) \in F\langle X\rangle$ is said to be multilinear if each variable x_i appears exactly once in each monomial of f.

Example

The standard polynomial $St_m = \sum_{\sigma \in S_m} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(m)}$ is multilinear.

Polynomial identities

Definition

A polynomial $p = p(x_1, \ldots, x_m) \in F\langle X \rangle$ is a polynomial identity of the algebra A if $p(a_1, \ldots, a_m) = 0$, for any $a_1, \ldots, a_m \in A$.

Theorem (Amitsur, Levitzki, 1950)

$St_{2d}(x_1, \ldots, x_{2d})$ is a polynomial identity of $M_d(F)$.

$M_d(F)$ has no identities of degree $< 2d$

Central polynomials

Definition

A polynomial $c = c(x_1, \dots, x_m) \in F\langle X\rangle$ is a central polynomial of A if c is not a polynomial identity of A and $c(a_1, \dots, a_m) \in Z(A)$, for any $a_1, \dots, a_m \in A$, where $Z(A)$ is the center of A.

Theorem (Formanek, 1972 - Razmyslov, 1973)

There exist central polynomials on $M_d(F)$.

Example

$[x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2]$ is a central polynomial on $M_2(F)$.

Open problem: minimal degree of central polynomials on matrices.

Preserving zeros of a polynomial

Let A be an algebra and let f be a multilinear polynomial.

Definition

An m-linear functional $\varphi\colon A^m \to F$ preserves zeros of f if

$$f(a_1, \dots, a_m) = 0 \implies \varphi(a_1, \dots, a_m) = 0.$$

Example

Let $\tau\colon A \to F$ be a linear functional. Then

$$\begin{aligned} \varphi\colon \quad A^m &\longrightarrow F \\ (a_1, \dots, a_m) &\longmapsto \tau(f(a_1, \dots, a_m)) \end{aligned}$$

preserves zeros of f.

f-zpd algebras

Definition

Let $f = f(x_1, \dots, x_m)$ be a multilinear polynomial. An algebra A is said to be f-zero product determined (f-zpd) if, for every m-linear functional $\varphi\colon A^m \to F$ preserving zeros of f, there exists a linear functional $\tau\colon A \to F$ such that

$$\varphi(a_1, \dots, a_m) = \tau\left(f(a_1, \dots, a_m)\right).$$

Remark If $f = x_1x_2$, then f-zpd is the same of zpd.

A counterexample

Theorem (Bajuk, Brešar, Fagundes, I., 2023)

$M_d(F)$ is not f-zpd, for any multilinear polynomial f.

Proof. Take $c(x_1, \dots, x_m)$ central multilinear polynomial of $M_d(F)$. One shows that $M_d(F)$ is not g-zpd, where

$$g = c(x_1, \dots, x_{m-1}, x_m)x_{m+1} + c(x_1, \dots, x_{m-1}, x_{m+1})x_m.$$

Positive results

Generalized commutator: $h = x_1 x_2 x_3 - x_3 x_2 x_1$.

Theorem (Bajuk, Brešar, Fagundes, I., 2023)

$M_d(F)$ is h-zpd.

Positive results

Take char$F \neq 2$. Let $\alpha_1, \ldots, \alpha_m \in F$ such that $\sum_{i=1}^{m} \alpha_i \neq 0$.

Polynomials given by cyclic permutations:

$$g = \alpha_1 x_1 \cdots x_m + \alpha_2 x_2 \cdots x_m x_1 + \cdots + \alpha_m x_m x_1 \cdots x_{m-1}.$$

Theorem (Bajuk, Brešar, Fagundes, I., 2023)

Let A be an F-algebra generated by idempotents. Then A is g-zpd.

$M_d(F)$ is generated by idempotents.

Positive results

Let $f_0, f_1, \dots, f_k$ be multilinear polynomials (f_0 in k variables).

Theorem (Bajuk, Brešar, Fagundes, I., 2023)

Let A be an F-algebra such that

(a) *A is f_i-zpd, $i = 0, 1, \dots, k$,*

(b) *$f_i(A) = A$, $i = 1, \dots, k$.*

Then A is f-zpd, where $f = f_0(f_1, \dots, f_k)$.

Remark Hypothesis (b) is not artificial:

Conjecture (L'vov-Kaplansky)

The image of a multilinear polynomial on $M_d(F)$ is a vector space.

Hilbert's Nullstellensatz

- $F[\xi_1, \ldots, \xi_m]$ ring of polynomials in commutative variables.
- $G = \{g(\xi_1, \ldots, \xi_m]) \in F[\xi_1, \ldots, \xi_m]\}$ set of polynomials.
- $I(G)$ ideal generated by G.

Theorem (Hilbert's Nullstellensatz)

Let $f \in F[\xi_1, \ldots, \xi_m]$ be a polynomial vanishing for all the zeros of the polynomials in the set G. Then $f^m \in I(G)$, for some $m \geq 1$.

Amitsur's Nullstellensatz

- $F\langle X\rangle$ ring of polynomials in non-commutative variables.
- $G = \{g(x_1, \ldots, x_m]) \in F\langle X\rangle\}$ set of polynomials.
- $I(G)$ ideal generated by G.
- M ideal generated by the commutators $x_i x_j - x_j x_i$.

Theorem (Amitsur's Nullstellensatz)

Assume that $f \in F\langle X\rangle$ satisfies the following property:

$$g(a_1, \ldots, a_m) = 0, \text{ for all } g \in G \Longrightarrow f(a_1, \ldots, a_m) = 0, \; a_i \in F.$$

Then, for some $m \geq 1$, $f^m \in I(G) \cup M$.

A multilinear Nullstellensatz

Theorem (Bajuk, Brešar, Fagundes, I., 2023)

Let f, g be multilinear polynomials of degree m. Assume that

$$g(a_1, \dots, a_m) = 0 \implies f(a_1, \dots, a_m) = 0, \quad a_i \in M_d(F).$$

- *If $M_d(F)$ is f-zpd and $f(1, \dots, 1) \neq 0$, $f - \lambda g \in \mathrm{Id}(M_d(F))$.*
- *If $m < 2d$, then $f = \lambda g$, $\lambda \in F$.*

A counterexample for $d = 2$ and $m = 5$

Example

Let $M_2(F)$ be the algebra of 2×2 matrices.

- $c(x_1, \dots, x_4) = [x_1, x_2][x_3, x_4] + [x_3, x_4][x_1, x_2]$.
- $f = c(x_1, \dots, x_4)x_5$.
- $g = c(x_1, \dots, x_4)x_5 + c(x_1, x_2, x_3, x_5)x_4$.

Remark

- $g(a_1, \dots, a_5) = 0 \implies f(a_1, \dots, a_5) = 0, \quad a_i \in M_2(F)$.
- $f - g = -c(x_1, x_2, x_3, x_5)x_4 \notin \mathrm{Id}(M_2(F))$.

Thank you for your attention.