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Heaps, trusses, Hopf heaps

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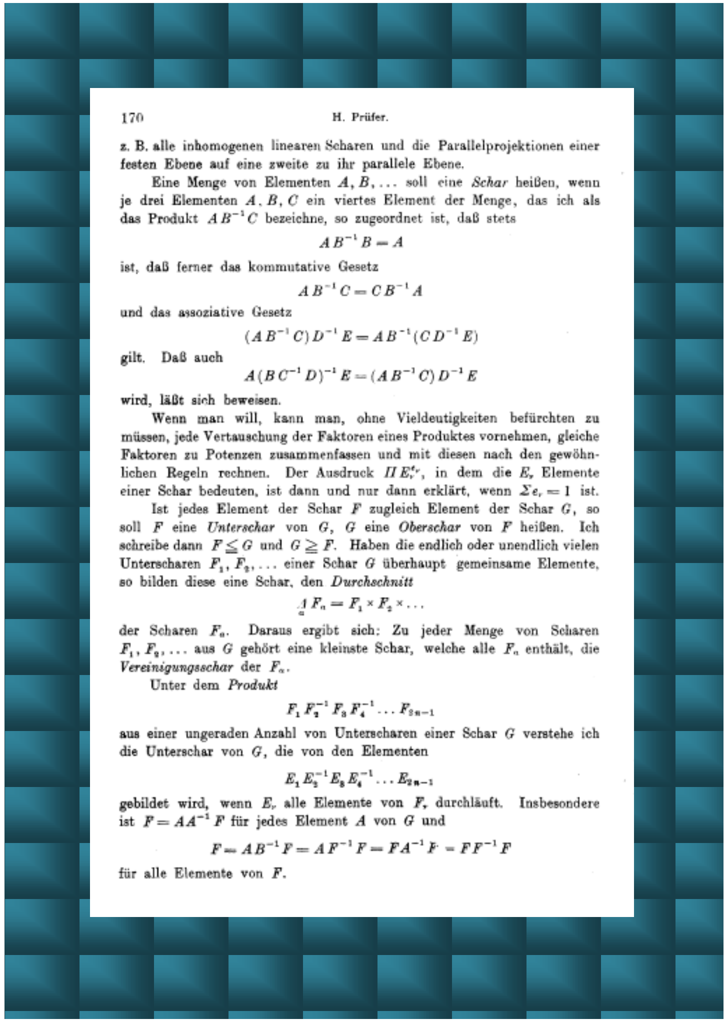
Heinz Prüfer [*Theorie der Abelschen Gruppen. I. Grundeigenschaften,* Mathematische Zeitschrift 20 (1924), 165–187, page 170].

Reinhold Baer [*Zur Einführung des Scharbegriffs,* Journal für die Reine und Angewandte Mathematik 160 (1929), 199–207, page 202].



170 H. Prüfer.

z. B. alle inhomogenen linearen Scharen und die Parallelprojektionen einer festen Ebene auf eine zweite zu ihr parallele Ebene.

Eine Menge von Elementen $A, B, \ldots$ soll eine *Schar* heißen, wenn je drei Elementen A, B, C ein viertes Element der Menge, das ich als das Produkt $AB^{-1}C$ bezeichne, so zugeordnet ist, daß stets

$$AB^{-1}B = A$$

ist, daß ferner das kommutative Gesetz

$$AB^{-1}C = CB^{-1}A$$

und das assoziative Gesetz

$$(AB^{-1}C)D^{-1}E = AB^{-1}(CD^{-1}E)$$

gilt. Daß auch

$$A(BC^{-1}D)^{-1}E = (AB^{-1}C)D^{-1}E$$

wird, läßt sich beweisen.

Wenn man will, kann man, ohne Vieldeutigkeiten befürchten zu müssen, jede Vertauschung der Faktoren eines Produktes vornehmen, gleiche Faktoren zu Potenzen zusammenfassen und mit diesen nach den gewöhnlichen Regeln rechnen. Der Ausdruck $\Pi E_\nu^{e_\nu}$, in dem die E_ν Elemente einer Schar bedeuten, ist dann und nur dann erklärt, wenn $\Sigma e_\nu = 1$ ist.

Ist jedes Element der Schar F zugleich Element der Schar G, so soll F eine *Unterschar* von G, G eine *Oberschar* von F heißen. Ich schreibe dann $F \leqq G$ und $G \geqq F$. Haben die endlich oder unendlich vielen Unterscharen $F_1, F_2, \ldots$ einer Schar G überhaupt gemeinsame Elemente, so bilden diese eine Schar, den *Durchschnitt*

$$\underset{\alpha}{\Delta} F_\alpha = F_1 \times F_2 \times \ldots$$

der Scharen F_α. Daraus ergibt sich: Zu jeder Menge von Scharen $F_1, F_2, \ldots$ aus G gehört eine kleinste Schar, welche alle F_α enthält, die *Vereinigungsschar* der F_α.

Unter dem *Produkt*

$$F_1 F_2^{-1} F_3 F_4^{-1} \ldots F_{2n-1}$$

aus einer ungeraden Anzahl von Unterscharen einer Schar G verstehe ich die Unterschar von G, die von den Elementen

$$E_1 E_2^{-1} E_3 E_4^{-1} \ldots E_{2n-1}$$

gebildet wird, wenn E_ν alle Elemente von F_ν durchläuft. Insbesondere ist $F = AA^{-1}F$ für jedes Element A von G und

$$F = AB^{-1}F = AF^{-1}F = FA^{-1}F = FF^{-1}F$$

für alle Elemente von F.

202 *Baer, Zur Einführung des Scharbegrifs.*

Setzen wir hier (5) ein, so ergibt sich:

$$(\mathfrak{z}_1 \mathfrak{x}_{i_1} \mathfrak{z}_2)^{e_2} (\mathfrak{z}_1 \mathfrak{x}_{i_1} \mathfrak{z}_2)^{e_3} J''(\cdot) = \mathfrak{z}_2^{-1} \mathfrak{x}_{i_1}^{e_2} \mathfrak{x}_{i_2}^{e_3} J''(\cdot)\, \mathfrak{z}_2$$

und hieraus folgt wie oben sukzessive:

$$e_2 = -1,\ e_3 = +1,$$

so daß sich:

$$J(\cdot) = \mathfrak{x}_{i_1} \mathfrak{x}_{i_2}^{-1} \mathfrak{x}_{i_3} J''(\cdot)$$

ergibt.

Analog weiter schließend erhält man:

(6) $$J(\cdot) = \mathfrak{x}_{i_1} \mathfrak{x}_{i_2}^{-1} \mathfrak{x}_{i_3} \mathfrak{x}_{i_4}^{-1} \mathfrak{x}_{i_5} \cdots \mathfrak{x}_{i_{m-1}}^{-1} \mathfrak{x}_{i_m},$$

wo stets $i_k \neq i_{k+1}$ und notwendig $m > 1$ und ungerade ist.

Sei $m = 2l + 1$, $l \geqq 1$.

Dann ist unser Satz bereits für $l = 1$ bewiesen; sei er auch für $l - 1$ bewiesen; es ist $J(\cdot) = \bar{J}(\cdot)\, \mathfrak{x}_{i_{m-1}}^{-1} \mathfrak{x}_{i_m}$, wo also n. V. $\bar{J}$ bereits rational und durch Einsetzen aus $F(\cdot)$ aufgebaut ist; dann ist:

$$J(\cdot) = F[\bar{J}(\cdot), \mathfrak{x}_{i_{m-1}}, \mathfrak{x}_{i_m}],$$

womit unser Satz bewiesen ist.

Bemerkung: Daß $F(\cdot)$ in speziellen Fällen kein vollständiges Invariantensystem ist, zeigt das folgende Beispiel:

$\mathfrak{G}$ sei *Abelsch* und werde durch $\mathfrak{a}_i$ mit $\mathfrak{a}_i^2 = 1$ $(i = 1, 2, 3, 4, 5)$ erzeugt. Dann wird auch

$$J[(\mathfrak{x}_i)] = \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 \mathfrak{x}_4$$

durch A isomorph abgebildet; denn die allgemeine ähnliche Abbildung ist:

$$\mathfrak{x} \to \alpha(\mathfrak{x}) = \mathfrak{x}\,\mathfrak{z};$$

also ist:

$$J[(\alpha(\mathfrak{x}_i))] = \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 \mathfrak{x}_4 \mathfrak{z}^4 = \mathfrak{x}_1 \mathfrak{x}_2 \mathfrak{x}_3 \mathfrak{x}_4 \mathfrak{z} = \alpha[J((\mathfrak{x}_i))].$$

und $J(\cdot)$ hat nicht die Form (6). Übrigens gilt unser Satz 2 auch für *Abelsche* Gruppen mit Elementen unendlich hoher Ordnung.

§ 2. Die Scharen.

Definition 2: *Eine Teilmenge $\mathfrak{S}$ der Gruppe $\mathfrak{G}$ heißt dann und nur dann eine Schar, wenn $\mathfrak{S}$ mit den Elementen $\mathfrak{a}_i (i = 1, 2, 3)$ stets auch das Element $\mathfrak{a}_1 \mathfrak{a}_2^{-1} \mathfrak{a}_3$ enthält.*

Satz 3: 1. *Eine Schar $\mathfrak{S}$ in $\mathfrak{G}$ ist dann und nur dann eine Untergruppe von $\mathfrak{G}$, wenn $\mathfrak{S}$ die Identität von $\mathfrak{G}$ enthält*[1]).

2. *Eine die Identität enthaltende Teilmenge $\mathfrak{S}$ von $\mathfrak{G}$ ist dann und nur dann eine Schar, wenn sie eine Untergruppe von $\mathfrak{G}$ ist.*

3. *Eine Teilmenge $\mathfrak{S}$ von $\mathfrak{G}$ ist dann und nur dann eine Schar in $\mathfrak{G}$, wenn es eine Untergruppe $\mathfrak{S}_r$ bezw. $\mathfrak{S}_l$ von $\mathfrak{G}$ gibt, so daß $\mathfrak{S}$ rechts- bezw. linksseitige Restklasse nach $\mathfrak{S}_r$ bezw. $\mathfrak{S}_l$ ist. $\mathfrak{S}_r$ bezw. $\mathfrak{S}_l$ sind hierdurch und durch $\mathfrak{S}$ eindeutig bestimmt.*

Beweis: I. Eine Teilmenge $\mathfrak{S}$ von $\mathfrak{G}$ ist dann und nur dann eine Untergruppe von $\mathfrak{G}$, wenn sie mit $\mathfrak{a}$ und $\mathfrak{b}$ auch $\mathfrak{a}^{-1}$ und $\mathfrak{a} \cdot \mathfrak{b}$ enthält; sie enthält also notwendig $\mathfrak{a}\mathfrak{a}^{-1} = 1$ und mit $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ auch $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}$.

[1]) Führt man den Scharbegriff abstrakt ein, wie es *Prüfer*, l. c. p. 170 tut, indem man unter einer Schar ein Elementesystem versteht, in dem je drei Elemente $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ eindeutig ein viertes $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}$ bestimmen, so daß I. $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{b} = \mathfrak{b}\mathfrak{b}^{-1}\mathfrak{a} = \mathfrak{a}$, II. $(\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c})\mathfrak{b}^{-1}\mathfrak{e} = \mathfrak{a}(\mathfrak{b}\mathfrak{c}^{-1}\mathfrak{b})^{-1}\mathfrak{e} = \mathfrak{a}\mathfrak{b}^{-1}(\mathfrak{c}\mathfrak{b}^{-1}\mathfrak{e}) = \mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c}\mathfrak{b}^{-1}\mathfrak{e}$, III. die Gl. $\mathfrak{a}\mathfrak{b}^{-1}\mathfrak{c} = \mathfrak{b}$ nach jedem der Elemente $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ auf nur eine Weise auflösbar ist — die Lösbarkeit überhaupt läßt sich beweisen —, so kann man von der Schar jederzeit zur Gruppe übergehen, indem man irgendein festes Element $\mathfrak{o}$ als Einheit auszeichnet und als Gruppenkomposition $\mathfrak{a} \circ \mathfrak{b} = \mathfrak{a}\mathfrak{o}^{-1}\mathfrak{b}$ einführt.

A heap is an algebraic system $(H, [-,-,-])$ consisting of a nonempty set H, and a ternary operation

$$[-,-,-]\colon H \times H \times H \to H, \quad (x,y,z) \mapsto [x,y,z]$$

satisfying

the heap associativity $[[x,y,z],t,u] = [x,y,[z,t,u]]$, $\neq [x,[y,z,t],u]$

Mal'cev identities $[x,x,y] = [y,x,x] = y$, $\neq [x,y,x]$

where $x,y,z,t,u \in H$. A heap $(H, [-,-,-])$ is abelian, if satisfies

the heap commutativity $[x,y,z] = [z,y,x]$,

where $x,y,z \in H$.

A heap homomorphism is a function $\varphi\colon (H, [-,-,-,]) \to (\widetilde{H}, [-,-,-,])$ respecting the heap operation

$$\varphi([x,y,z]) = [\varphi(x), \varphi(y), \varphi(z)],$$

where $x,y,z \in H$.

Theorem. Let $(H, [-,-,-])$ be a heap.

(a) For any $x, y, z \in H$,

$$[y, x, [x, y, z]] = z, \qquad [[x, y, z], z, y] = x.$$

Indeed,

$$[y, x, [x, y, z]] = [[y, x, x], y, z] = [y, y, z] = z$$

and

$$[[x, y, z], z, y] = [x, y, [z, z, y]] = [x, y, y] = x.$$




(b) For any $x, y, z, t, u \in H$,

$$[x, y, [z, t, u]] = [x, [t, z, y], u].$$

Indeed, since

$$[[t, z, y], x, [x, y, [z, t, u]]] = [[[t, z, y], x, x], y, [z, t, u]] =$$
$$= [[t, z, y], y, [z, t, u]] = [[[t, z, y], y, z], t, u] = [t, t, u] = u$$

and

$$[[t, z, y], x, [x, [t, z, y], u]] = u,$$

it follows that

$$[[t, z, y], x, [x, y, [z, t, u]]] = [[t, z, y], x, [x, [t, z, y], u]],$$

and thus

$$[x, y, [z, t, u]] = [x, [t, z, y], [[t, z, y], x, [x, y, [z, t, u]]]] =$$
$$= [x, [t, z, y], [[t, z, y], x, [x, [t, z, y], u]]] = [x, [t, z, y], u].$$




(c) For any $x, y, z \in H$,

$$[z, [x, y, z], x] = y.$$

Indeed,

$[z, [x, y, z], x] = [z, z, [y, x, x]] = [z, z, y] = y.$




(d) $(H, [-,-,-])$ is an abelian heap iff for any $x, y, z, t, u \in H$,

$$[x, y, [z, t, u]] = [x, [y, z, t], u].$$

Indeed, if $(H, [-,-,-])$ is abelian, then

$[x, y, [z, t, u]] = [x, [t, z, y], u] = [x, [y, z, t], u]$.

If for any $x, y, z, t, u \in H$,

$[x, y, [z, t, u]] = [x, [y, z, t], u]$,

then since also

$[x, y, [z, t, u]] = [x, [t, z, y], u]$,

it follows that

$[x, [y, z, t], u] = [x, [t, z, y], u]$,

and thus

$[y, z, t] = [u, [x, [y, z, t], u], x] = [u, [x, [t, z, y], u], x] = [t, z, y]$. □




Theorem. Given a group $(G, \circ, 1)$, let

$$[-,-,-]_\circ \colon G \times G \times G \to G, \quad [x,y,z]_\circ := x \circ y^{-1} \circ z,$$

where $x, y, z \in G$. Then

(a) $(G, [-,-,-]_\circ)$ is a heap.

Indeed, for any $x, y, z, t, u \in G$,

$$[[x,y,z]_\circ, t, u]_\circ = (x \circ y^{-1} \circ z) \circ t^{-1} \circ u = x \circ y^{-1} \circ (z \circ t^{-1} \circ u) = [x, y, [z,t,u]_\circ]_\circ$$

$$[x,x,y]_\circ = x \circ x^{-1} \circ y = y = y \circ x^{-1} \circ x = [y,x,x]_\circ.$$

(b) If $(G, \circ, 1)$ is an abelian group, then $(G, [-,-,-]_\circ)$ is an abelian heap.

Indeed, for any $x, y, z \in G$,

$$[x,y,z]_\circ = x \circ y^{-1} \circ z = z \circ y^{-1} \circ x = [z,y,x]_\circ.$$

(c) Every group homomorphism $\varphi \colon (G, \circ, 1) \to (\widetilde{G}, \circ, 1)$

is an associated heap homomorphism $\varphi \colon (G, [-,-,-]_\circ) \to (\widetilde{G}, [-,-,-]_\circ)$.

Indeed, for any $x, y, z \in G$,

$$\varphi([x,y,z]_\circ) = \varphi(x \circ y^{-1} \circ z) = \varphi(x) \circ \varphi(y)^{-1} \circ \varphi(z) = [\varphi(x), \varphi(y), \varphi(z)]_\circ. \qquad \square$$

Theorem. Given a heap $(H, [-,-,-])$ and $e \in H$, let

$$\circ_e \colon H \times H \to H, \qquad x \circ_e y := [x, e, y],$$

where $x, y \in H$. Then

(a) $(H, \circ_e, e)$ is a group, known as a retract of $(H, [-,-,-])$.

Indeed, for any $x, y, z \in H$,

$(x \circ_e y) \circ_e z = [[x, e, y], e, z] = [x, e, [y, e, z]] = x \circ_e (y \circ_e z)$

$e \circ_e x = [e, e, x] = x = [x, e, e] = x \circ_e e$

$$\begin{aligned}[e, x, e] \circ_e x &= [[e, x, e], e, x] = [e, x, [e, e, x]] = [e, x, x] = e = \\ &= [x, x, e] = [[x, e, e], x, e] = [x, e, [e, x, e]] = x \circ_e [e, x, e],\end{aligned}$$

so $x^{-1} = [e, x, e]$.

(b) If $(H, [-,-,-])$ is an abelian heap, then $(H, \circ_e, e)$ is an abelian group.

Indeed, for any $x, y \in H$,

$x \circ_e y = [x, e, y] = [y, e, x] = y \circ_e x.$

(c) If $\varphi\colon (H,[-,-,-]) \to (\widetilde{H},[-,-,-])$ is a heap homomorphism,

then for any $e \in H$, $\tilde{e} \in \widetilde{H}$, the functions

$$\widehat{\varphi}\colon (H,\circ_e,e) \to (\widetilde{H},\circ_{\tilde{e}},\tilde{e}), \qquad x \mapsto [\varphi(x),\varphi(e),\tilde{e}]$$

$$\widehat{\varphi}^\circ\colon (H,\circ_e,e) \to (\widetilde{H},\circ_{\tilde{e}},\tilde{e}), \qquad x \mapsto [\tilde{e},\varphi(e),\varphi(x)]$$

are associated group homomorphisms.

Indeed, for any $x,y \in H$,

$$\begin{aligned}\widehat{\varphi}(x\circ_e y) &= [\varphi(x\circ_e y),\varphi(e),\tilde{e}] = [\varphi([x,e,y]),\varphi(e),\tilde{e}] = \\ &= [[\varphi(x),\varphi(e),\varphi(y)],\varphi(e),\tilde{e}] = [\varphi(x),\varphi(e),[\varphi(y),\varphi(e),\tilde{e}]] = \\ &= [\varphi(x),\varphi(e),\widehat{\varphi}(y)] = [\varphi(x),\varphi(e),[\tilde{e},\tilde{e},\widehat{\varphi}(y)]] = \\ &= [[\varphi(x),\varphi(e),\tilde{e}],\tilde{e},\widehat{\varphi}(y)] = [\widehat{\varphi}(x),\tilde{e},\widehat{\varphi}(y)] = \widehat{\varphi}(x)\circ_{\tilde{e}}\widehat{\varphi}(y).\end{aligned}$$

In a similar manner, $\widehat{\varphi}^\circ(x\circ_e y) = \widehat{\varphi}^\circ(x)\circ_{\tilde{e}}\widehat{\varphi}^\circ(y)$. □

Corollary. Given a heap $(H,[-,-,-])$ and $e,f \in H$,

let $(H,\circ_e,e)$ and $(H,\circ_f,f)$ be the groups associated to the heap $(H,[-,-,-])$.

Then the function

$$\tau_e^f\colon (H,\circ_e,e) \to (H,\circ_f,f), \quad x \mapsto [x,e,f]$$

is an associated group isomorphism with the inverse $(\tau_e^f)^{-1} = \tau_f^e$.

Indeed, the function

$\tau_e^f := \widehat{\mathrm{id}_H}\colon (H,\circ_e,e) \to (H,\circ_f,f), \quad x \mapsto [\mathrm{id}_H(x),\mathrm{id}_H(e),f] = [x,e,f]$

is a group isomorphism. □




A group $(G, \circ, 1)$

$\Downarrow$

The heap $(G, [-,-,-]_\circ)$ associated to the group $(G, \circ, 1)$,

where $[x, y, z]_\circ := x \circ y^{-1} \circ z$

$\Downarrow$

$\forall e \in G$, The group $(G, \circ_e, e)$ associated to the heap $(G, [-,-,-]_\circ)$,

where $x \circ_e y := [x, e, y]_\circ = x \circ e^{-1} \circ y$




A heap $(H, [-,-,-])$

$\Downarrow$

$\forall e \in H$, The group $(H, \circ_e, e)$ associated to the heap $(H, [-,-,-])$,

where $x \circ_e y := [x, e, y]$

$\Downarrow$

The heap $(H, [-,-,-]_{\circ_e})$ associated to the group $(H, \circ_e, e)$,

where $[x, y, z]_{\circ_e} := x \circ_e y^{-1} \circ_e z = [[x, e, y^{-1}], e, z] = [[x, e, [e, y, e]], e, z]$




Theorem. Given a group $(G, \circ, 1)$ and $e \in G$,

let $(G, [-,-,-]_\circ)$ be the heap associated to the group $(G, \circ, 1)$,

let $(G, \circ_e, e)$ be the group associated to the heap $(G, [-,-,-]_\circ)$.

Then $(G, \circ, 1) \cong (G, \circ_e, e)$ as groups.

In particular, $\circ = \circ_1$.

Indeed, let $\varphi \colon (G, \circ, 1) \to (G, \circ_e, e)$, $\quad x \mapsto x \circ e$. Then for any $x, y \in G$,

$$\varphi(x \circ y) = (x \circ y) \circ e = (x \circ e) \circ e^{-1} \circ (y \circ e) =$$
$$= \varphi(x) \circ e^{-1} \circ \varphi(y) = [\varphi(x), e, \varphi(y)]_\circ = \varphi(x) \circ_e \varphi(y).$$

Hence φ is a group isomorphism with the inverse $\varphi^{-1} \colon (G, \circ_e, e) \to (G, \circ, 1)$, $\quad x \mapsto x \circ e^{-1}$.

$x \circ y = x \circ 1^{-1} \circ y = [x, 1, y]_\circ = x \circ_1 y$. □

Corollary (see slide 11). Given a group $(G, \circ, 1)$ and $e, f \in G$,

let $(G, [-,-,-]_\circ)$ be the heap associated to the group $(G, \circ, 1)$,

let $(G, \circ_e, e)$ and $(G, \circ_f, f)$ be the groups associated to the heap $(G, [-,-,-]_\circ)$.

Then the function

$$\tau_e^f \colon (G, \circ_e, e) \to (G, \circ_f, f), \quad x \mapsto x \circ e^{-1} \circ f$$

is an associated group isomorphism with the inverse $(\tau_e^f)^{-1} = \tau_f^e$.

Indeed,
let $\varphi \colon (G, \circ, 1) \to (G, \circ_e, e)$, $x \mapsto x \circ e$
and $\psi \colon (G, \circ, 1) \to (G, \circ_f, f)$, $x \mapsto x \circ f$.

Then $\tau_e^f = \psi\varphi^{-1}$. □




Theorem. Given a heap $(H, [-,-,-])$ and $e \in H$,

let $(H, \circ_e, e)$ be the group associated to the heap $(H, [-,-,-])$,

let $(H, [-,-,-]_{\circ_e})$ be the heap associated to the group $(H, \circ_e, e)$.

Then $[-,-,-] = [-,-,-]_{\circ_e}$.

Indeed, for any $x, y, z \in H$,

$$
\begin{aligned}
[x,y,z] &= [x,y,[e,e,z]] = [[x,y,e],e,z] = \\
&= [[[x,e,e],y,e],e,z] = [[x,e,[e,y,e]],e,z] = \\
&= [[x,e,y^{-1}],e,z] = x \circ_e y^{-1} \circ_e z = [x,y,z]_{\circ_e}.
\end{aligned}
$$

□




A Hopf heap is an algebraic system $(C, \Delta, \varepsilon, [-,-,-])$ consisting of

a vector space C over the field $\mathbb{F}$, and linear maps

$$\Delta\colon C \to C \otimes C, \qquad x \mapsto \sum x_1 \otimes x_2$$

$$\varepsilon\colon C \to \mathbb{F}$$

$$[-,-,-]\colon C \otimes C^{co} \otimes C \to C, \qquad x \otimes y \otimes z \mapsto [x, y, z],$$

such that

(C, Δ, ε) is a coalgebra

$$\sum \Delta(x_1) \otimes x_2 = \sum x_1 \otimes \Delta(x_2) = \sum x_1 \otimes x_2 \otimes x_3$$

$$\sum \varepsilon(x_1)x_2 = \sum x_1\varepsilon(x_2) = x,$$

$[-,-,-]\colon C \otimes C^{co} \otimes C \to C$ is a coalgebra map

$$\Delta([x,y,z]) = \sum [x_1, y_2, z_1] \otimes [x_2, y_1, z_2]$$

$$\varepsilon([x,y,z]) = \varepsilon(x)\varepsilon(y)\varepsilon(z),$$

satisfying

the heap associativity $\quad [[x,y,z],t,u] = [x,y,[z,t,u]]$

Mal'cev identities $\quad \sum [x_1, x_2, y] = \sum [y, x_1, x_2] = \varepsilon(x)y,$

where $x, y, z, t, u \in C$.

A homomorphism of Hopf heaps $(C, \Delta, \varepsilon, [-,-,-,]) \to (D, \Delta, \varepsilon, [-,-,-,])$ is

a coalgebra map $\varphi\colon (C, \Delta, \varepsilon) \to (D, \Delta, \varepsilon)$

$$\Delta(\varphi(x)) = \sum \varphi(x)_1 \otimes \varphi(x)_2 = \sum \varphi(x_1) \otimes \varphi(x_2)$$

$$\varepsilon(\varphi(x)) = \varepsilon(x)$$

respecting the heap operation

$$\varphi([x, y, z]) = [\varphi(x), \varphi(y), \varphi(z)],$$

where $x, y, z \in C$.




A Grunspan map for a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$ is a coalgebra map $\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$ such that

$$[[x, y, \theta(z)], t, u] = [x, [t, z, y], u],$$

where $x, y, z, t, u \in C$.

If it exists, the Grunspan map for a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$ is given by the formula

$$\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon), \qquad x \mapsto \sum [x_1, [x_4, x_3, x_2], x_5],$$

where $x \in C$, and thus necessarily is unique.




The Grunspan map commutes with every homomorphism of Hopf heaps,

that it, if $\varphi\colon (C, \Delta, \varepsilon, [-,-,-]) \to (D, \Delta, \varepsilon, [-,-,-])$

is a homomorphism of Hopf heaps with respective Grunspan maps

$\theta_C\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$ and $\theta_D\colon (D, \Delta, \varepsilon) \to (D, \Delta, \varepsilon)$, then

$$\varphi\theta_C = \theta_D\varphi.$$

Indeed, for any $x \in C$,

$$\varphi\theta_C(x) = \varphi(\sum[x_1, [x_4, x_3, x_2], x_5)] = \sum[\varphi(x_1), [\varphi(x_4), \varphi(x_3), \varphi(x_2)], \varphi(x_1)] =$$

$$= \sum[\varphi(x)_1, [\varphi(x)_4, \varphi(x)_3, \varphi(x)_2], \varphi(x)_5] = \theta_D\varphi(x).$$




Theorem. Every Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$ admits the Grunspan map $\theta\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$.

See [Tomasz Brzeziński, MH, *Translation Hopf algebras and Hopf heaps,*

arXiv: 2303.13154v1, Corollary 3.9]. □




Theorem. Given a Hopf algebra $(H, \Delta, \varepsilon, S)$, let

$$[-,-,-]_\bullet \colon H \otimes H^{co} \otimes H, \qquad [x,y,z]_\bullet := x \cdot S(y) \cdot z$$

$$\theta \colon H \to H, \qquad \theta(x) := S^2(x),$$

where $x, y, z \in H$. Then

(a) $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$ is a Hopf heap with the Grunspan map

$\theta \colon (H, \Delta, \varepsilon) \to (H, \Delta, \varepsilon)$.

Indeed, for any $x, y, z, t, u \in H$,

$$\begin{aligned}\Delta([x,y,z]_\bullet) &= \Delta(x \cdot S(y) \cdot z) = \Delta(x) \cdot \Delta(S(y)) \cdot \Delta(z) = \\ &= \sum (x_1 \otimes x_2) \cdot (S(y)_1 \otimes S(y)_2) \cdot (z_1 \otimes z_2) = \\ &= \sum (x_1 \otimes x_2) \cdot (S(y_2) \otimes S(y_1)) \cdot (z_1 \otimes z_2) = \\ &= \sum x_1 \cdot S(y_2) \cdot z_1 \otimes x_2 \cdot S(y_1) \cdot z_2 = \sum [x_1, y_2, z_1] \otimes [x_2, y_1, z_2]\end{aligned}$$






$$\varepsilon([x,y,z]_\bullet) = \varepsilon(x\cdot S(y)\cdot z) = \varepsilon(x)\varepsilon(S(y))\varepsilon(z) = \varepsilon(x)\varepsilon(y)\varepsilon(z)$$

$$[[x,y,z]_\bullet,t,u]_\bullet = (x\cdot S(y)\cdot z)\cdot S(t)\cdot u = x\cdot S(y)\cdot(z\cdot S(t)\cdot u) = [x,y,[z,y,u]_\bullet]_\bullet$$

$$\sum[x_1,x_2,y]_\bullet = \sum x_1\cdot S(x_2)\cdot y = \varepsilon(x)y = \sum y\cdot S(x_1)\cdot x_2 = \sum[y,x_1,x_2]_\bullet$$

$$[[x,y,\theta(z)]_\bullet,t,u]_\bullet = x\cdot S(y)\cdot S^2(z)\cdot S(t)\cdot u = x\cdot S(t\cdot S(z)\cdot y)\cdot u = [x,[t,z,y]_\bullet,u]_\bullet.$$

(b) Every homomorphism of Hopf algebras $\varphi\colon (H,\Delta,\varepsilon,S_H)\to(\widetilde{H},\Delta,\varepsilon,S_{\widetilde{H}})$

is a homomorphism of associated Hopf heaps

$\varphi\colon (H,\Delta,\varepsilon,[-,-,-]_\bullet)\to(\widetilde{H},\Delta,\varepsilon,[-,-,-]_\bullet)$.

Indeed, for any $x,y,z\in H$,

$$\varphi([x,y,z]_\bullet) = \varphi(x\cdot S(y)\cdot z) = \varphi(x)\cdot\varphi(S_H(y))\cdot\varphi(z) =$$
$$= \varphi(x)\cdot S_{\widetilde{H}}(\varphi(y))\cdot\varphi(z) = [\varphi(x),\varphi(y),\varphi(z)]_\bullet. \qquad \square$$




Theorem. Given a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

and a group-like element $e \in G(C)$, let

$$\cdot_e \colon C \otimes C, \qquad x \cdot_e y := [x, e, y]$$

$$S_e \colon C \to C, \qquad S_e(x) := [e, x, e]$$

where $x, y \in C$. Then

(a) $(C, \Delta, \varepsilon, \cdot_e, e, S_e)$ is a Hopf algebra denoted by $H_e(C)$.

Indeed, for any $x, y, z \in C$,

$$(x \cdot_e y) \cdot_e z = [[x, e, y], e, z] = [x, e, [y, e, z]] = x \cdot_e (y \cdot_e z)$$

$$e \cdot_e x = [e, e, x] = x = [x, e, e] = x \cdot_e e$$




$$\Delta(x \cdot_e y) = \Delta([x, e, y]) = \sum [x_1, e, y_1] \otimes [x_2, e, y_2] =$$

$$= \sum x_1 \cdot_e y_1 \otimes x_2 \cdot_e y_2 = \sum (x_1 \otimes x_2) \cdot_e (y_1 \otimes y_2) = \Delta(x) \cdot_e \Delta(y)$$

$$\Delta(e) = e \otimes e$$

$$\varepsilon(x \cdot_e y) = \varepsilon([x, e, y]) = \varepsilon(x)\varepsilon(e)\varepsilon(y) = \varepsilon(x)\varepsilon(y)$$

$$\varepsilon(e) = 1$$

$$\sum S_e(x_1) \cdot_e x_2 = \sum [[e, x_1, e], e, x_2] = \sum [e, x_1, [e, e, x_2]] = \sum [e, x_1, x_2] = \varepsilon(x)e$$

$$\sum x_1 \cdot_e S_e(x_2) = \sum [x_1, e, [e, x_2, e]] = \sum [[x_1, e, e], x_2, e] = \sum [x_1, x_2, e] = \varepsilon(x)e.$$




(b) If $\varphi\colon (C,\Delta,\varepsilon,[-,-,-]) \to (D,\Delta,\varepsilon,[-,-,-])$ is a homomorphism of Hopf heaps, then for any $e \in G(C)$ and $f \in G(D)$, the functions

$$\widehat{\varphi}\colon (H_e(C),\Delta,\varepsilon,\cdot_e,e) \to (H_f(D),\Delta,\varepsilon,\cdot_f,f), \quad x \mapsto [\varphi(x),\varphi(e),f]$$

$$\widehat{\varphi}^\circ\colon (H_e(C),\Delta,\varepsilon,\cdot_e,e) \to (H_f(D),\Delta,\varepsilon,\cdot_f,f), \quad x \mapsto [f,\varphi(e),\varphi(x)]$$

are associated bialgebra homomorphisms.

Indeed, for any $x,y \in C$,

$$\begin{aligned}\widehat{\varphi}(x \cdot_e y) = \widehat{\varphi}([x,e,y]) = [\varphi([x,e,y]),\varphi(e),f] = [[\varphi(x),\varphi(e),\varphi(y)],\varphi(e),f] =\\ = [\varphi(x),\varphi(e),[\varphi(y),\varphi(e),f]] = [\varphi(x),\varphi(e),\widehat{\varphi}(y)] =\\ = [\varphi(x),\varphi(e),[f,f,\widehat{\varphi}(y)]] = [[\varphi(x),\varphi(e),f],f,\widehat{\varphi}(y)] = [\widehat{\varphi}(x),f,\widehat{\varphi}(y)] = \widehat{\varphi}(x)\cdot_f \widehat{\varphi}(y)\end{aligned}$$

Since $\varphi\colon (C,\Delta,\varepsilon) \to (D,\Delta,\varepsilon)$ is a coalgebra map and $e \in G(C)$ is a group-like element, it follows that

$$\Delta(\varphi(e)) = \sum \varphi(e)_1 \otimes \varphi(e)_2 = \varphi(e)\otimes\varphi(e), \quad \varepsilon(\varphi(e)) = \varepsilon(e) = 1_{\mathbb{F}}.$$




This means that $\varphi(e) \in G(D)$ is a group-like element, and hence

$$\widehat{\varphi}(e) = [\varphi(e), \varphi(e), f] = f$$

$$\begin{aligned}\Delta(\widehat{\varphi}(x)) &= \sum \widehat{\varphi}(x)_1 \otimes \widehat{\varphi}(x)_2 = \\ &= \sum [\varphi(x), \varphi(e), f]_1 \otimes [\varphi(x), \varphi(e), f]_2 = \\ &= \sum [\varphi(x)_1, \varphi(e)_2, f] \otimes [\varphi(x)_2, \varphi(e)_1, f] = \\ &= \sum [\varphi(x_1), \varphi(e), f] \otimes [\varphi(x_2), \varphi(e), f] = \sum \widehat{\varphi}(x_1) \otimes \widehat{\varphi}(x_2)\end{aligned}$$

$$\varepsilon(\widehat{\varphi}(x)) = \varepsilon([\varphi(x), \varphi(e), f]) = \varepsilon(\varphi(x))\varepsilon(\varphi(e))\varepsilon(f) = \varepsilon(x).$$




(c) If both the Hopf heaps $(C, \Delta, \varepsilon, [-,-,-])$ and $(D, \Delta, \varepsilon, [-,-,-])$ admit the Grunspan maps $\theta_C\colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$ and $\theta_D\colon (D, \Delta, \varepsilon) \to (D, \Delta, \varepsilon)$, then both the maps

$$\widehat{\varphi}\colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f(D), \Delta, \varepsilon, \cdot_f, f), \quad x \mapsto [\varphi(x), \varphi(e), f]$$

$$\widehat{\varphi}^\circ\colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f(D), \Delta, \varepsilon, \cdot_f, f), \quad x \mapsto [f, \varphi(e), \varphi(x)]$$

are homomorphisms of associated Hopf algebras.

Indeed, for any $x \in C$,

$$\begin{aligned}
S_f(\widehat{\varphi}(x)) &= [f, \widehat{\varphi}(x), f] = [f, [\varphi(x), \varphi(e), f], f] = \\
&= [[f, f, \theta_D(\varphi(e))], \varphi(x), f] = [\theta_D(\varphi(e)), \varphi(x), f] = \\
&= [\varphi(\theta_C(e)), \varphi(x), f] = [\varphi(e), \varphi(x), f] = \\
&= [\varphi(e), \varphi(x), [\varphi(e), \varphi(e), f]] = [[\varphi(e), \varphi(x), \varphi(e)], \varphi(e), f] = \\
&= [\varphi([e, x, e]), \varphi(e), f] = \widehat{\varphi}([e, x, e]) = \widehat{\varphi}(S_e(x)).
\end{aligned}$$

□

Corollary. Given a Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

and a group-like elements $e, f \in G(C)$,

let $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$ and $(H_f(C), \Delta, \varepsilon, \cdot_f, f, S_f)$ be the Hopf algebras

associated to the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$.

Then the function

$$\tau_e^f \colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f(C), \Delta, \varepsilon, \cdot_f, f), \quad x \mapsto [x, e, f]$$

is an associated bialgebra isomorphism with the inverse $(\tau_e^f)^{-1} = \tau_f^e$.

If the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$ admits

the Grunspan map $\theta \colon (C, \Delta, \varepsilon) \to (C, \Delta, \varepsilon)$,

then this function is an isomorphism of associated Hopf algebras.

Indeed, the function

$$\tau_e^f := \widehat{\mathrm{id}_C}\colon (H_e(C), \Delta, \varepsilon, \cdot_e, e) \to (H_f, \Delta, \varepsilon, \cdot_f, f), \quad x \mapsto [\mathrm{id}_C(x), \mathrm{id}_C(e), f] = [x, e, f]$$

is a bialgebra isomorphism. □

A Hopf algebra $(H, \Delta, \varepsilon, S)$

$\Downarrow$

The Hopf heap $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$ associated to

the Hopf algebra $(H, \Delta, \varepsilon, S)$,

where $[x, y, z]_\bullet := x \cdot S(y) \cdot z$

$\Downarrow$

$\forall e \in G(H)$, The Hopf algebra $(H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e)$ associated to

the Hopf heap $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$,

where $x \cdot_e y := [x, e, y]_\bullet = x \cdot S(e) \cdot y$

$$S_e(x) := [e, x, e]_\bullet = e \cdot S(x) \cdot e$$

A Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$

$\Downarrow$

$\forall e \in G(C)$, The Hopf algebra $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$ associated to

the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$,

where $x \cdot_e y := [x, e, y]$

$S_e(x) := [e, x, e]$

$\Downarrow$

The heap $(H_e(C), \Delta, \varepsilon, [-,-,-]_{\bullet_e})$ associated to

the Hopf algebra $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$,

where $[x, y, z]_{\bullet_e} := x \cdot_e S(y) \cdot_e z = [[x, e, S(y)], e, z] = [[x, e, [e, y, e]], e, z]$

Theorem. Given a Hopf algebra $(H, \Delta, \varepsilon, S)$ and a group-like element $e \in G(H)$,

let $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$ be the Hopf heap associated to

the Hopf algebra $(H, \Delta, \varepsilon, S)$,

let $(H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e)$ be the Hopf algebra associated to

the Hopf heap $(H, \Delta, \varepsilon, [-,-,-]_\bullet)$.

Then $(H, \Delta, \varepsilon, S) \cong (H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e)$ as Hopf algebras.

In particular $\cdot = \cdot_1$.

Indeed, let $\varphi\colon H \to H_e(H), \quad x \mapsto x \cdot e$. Then for any $x, y \in H$,

$\varphi(x \cdot y) = (x \cdot y) \cdot e = (x \cdot e) \cdot e^{-1} \cdot (y \cdot e) = \varphi(x) \cdot S(e) \cdot \varphi(y) = [\varphi(x), e, \varphi(y)]_\bullet = \varphi(x) \cdot_e \varphi(y)$

$\varphi(1) = 1 \cdot e = e$

$\Delta(\varphi(x)) = \Delta(x \cdot e) = \Delta(x) \cdot \Delta(e) = \sum (x_1 \otimes x_2) \cdot (e \otimes e) = \sum x_1 \cdot e \otimes x_2 \cdot e = \sum \varphi(x_1) \otimes \varphi(x_2)$

$\varepsilon(\varphi(x)) = \varepsilon(x \cdot e) = \varepsilon(x)\varepsilon(e) = \varepsilon(x)$

$$\varphi(S(x)) = S(x) \cdot e = e \cdot e^{-1} \cdot S(x) \cdot e = e \cdot S(e) \cdot S(x) \cdot e =$$

$$= e \cdot S(x \cdot e) \cdot e = e \cdot S(\varphi(x)) \cdot e = [e, \varphi(x), e]_{\bullet} = S_e(\varphi(x)).$$

Hence φ is an isomorphism of Hopf algebrs with the inverse $\varphi^{-1}\colon (H_e(H), \Delta, \varepsilon, \cdot_e, e, S_e) \to (H, \Delta, \varepsilon, S), \quad x \mapsto x \circ e^{-1}$.

$x \cdot y = x \cdot 1 \cdot y = x \cdot S(1) \cdot y = [x, 1, y]_{\bullet} = x \cdot_1 y$. $\square$




Theorem. Given a Hopf heap $(H, \Delta, \varepsilon, [-,-,-])$

and a group-like element $e \in G(C)$,

let $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$ be the Hopf algebra associated to

the Hopf heap $(C, \Delta, \varepsilon, [-,-,-])$,

let $(H_e(C), \Delta, \varepsilon, [-,-,-]_{\bullet_e})$ be the Hopf heap associated to

the Hopf algebra $(H_e(C), \Delta, \varepsilon, \cdot_e, e, S_e)$.

Then $[-,-,-] = [-,-,-]_{\bullet_e}$.

Indeed, for any $x, y, z \in C$,

$$
\begin{aligned}
[x,y,z] &= [x,y,[e,e,z]] = [[x,y,e],e,z] = \\
&= [[[x,e,e],y,e],e,z] = [[x,e,[e,y,e]],e,z] = \\
&= [[x,e,S(y)],e,z] = x \cdot_e S(y) \cdot_e z = [x,y,z]_{\bullet_e}.
\end{aligned}
$$

□






[Tomasz Brzeziński, MH, *Translation Hopf algebras and Hopf heaps,*

arXiv: 2303.13154v1]

Definition 2.4. Let $(C, \Delta, \varepsilon, [-,-,-])$ be a Hopf heap.

For any $a, b \in C$, the linear map

$$\tau_a^b \colon C \to C, \quad x \mapsto [x, a, b]$$

is called a right (a, b)-translation.

The space spanned by all right (a, b)-translations is denoted by $\mathsf{Tn}(C)$, that is,

$$\mathsf{Tn}(C) := \mathbb{F}\langle \tau_a^b \mid a, b \in C \rangle.$$

Symmetrically, linear maps

$$\sigma_a^b \colon C \to C, \quad x \mapsto [a, b, x]$$

are called left (a, b)-translations

and the space spanned by all of them is denoted by $\widehat{\mathsf{Tn}}(C)$.

Theorem 2.6. Let $(C, [-,-,-])$ be a Hopf heap. Then

(a) The space $\mathrm{Tn}(C)$ is a bialgebra with multiplication given by

the opposite composition, with comultiplication

$$\Delta(\tau_a^b) := \sum \tau_{a_2}^{b_1} \otimes \tau_{a_1}^{b^2}$$

and counit

$$\varepsilon(\tau_a^b) := \varepsilon(a)\varepsilon(b).$$

(b) If $(C[-,-,-])$ admits the Grunspan map θ,

then $\mathrm{Tn}(C)$ is a Hopf algebra with the antipode

$$S(\tau_a^b) := \tau_b^{\theta(a)}.$$

(c) If $f\colon C \to D$ is a homomorphism of Hopf heaps, then the function

$$\mathsf{Tn}(f)\colon \mathsf{Tn}(C) \to \mathsf{Tn}(D), \quad \tau_a^b \mapsto \tau_{f(a)}^{f(b)}$$

is a bialgebra map,

hence a Hopf algebra homomorphism whenever the Grunspan map exists.

(d) The assignment

$$C \mapsto \mathsf{Tn}(C), \quad f \mapsto \mathsf{Tn}(f)$$

defines a functor from the category of Hopf heaps (with Grunspan maps)

to the category of bialgebras (resp. Hopf algebras).




Corollary 3.9. Every Hopf heap admits the Grunspan map.




Thank you very much for your attention!

La ringrazio molto per l'attenzione!