The Structure of Metahamiltonian Groups

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Some examples:

- Dedekind groups (i.e. groups in which every subgroup is normal);
- Minimal non-abelian groups;
- $SL_2(3) \simeq \mathbb{Z}_3 \ltimes Q_8$.

The (scattered) structure of metahamiltonian groups

N.F. Kuzennyĭ – N.N. Semko: "The structure of solvable nonnilpotent meta-Hamiltonian groups", *Math. Notes* 34 (1983).

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L. An – Q. Zhang: "Finite metahamiltonian p-groups", *J. Algebra* 442 (2015).

Getting a grip on metahamiltonian groups

Dedekind groups

R. Dedekind – 1897

Every subgroup of a group G is abelian if and only if

(1) G is abelian or

(2) $G = Q_8 \times A \times B$, where A is an elementary abelian 2-group and B is a torsion abelian group with no element of even order.

Getting a grip on metahamiltonian groups

Minimal non-abelian p-groups

L. Redei – 1947

Let G be a minimal non-abelian p-group for some prime p. Then $|G'| = p$, G/G' is abelian of rank two and G is one of the following groups

(1) $G = \langle a, b \,|\, a^{p^m} = b^{p^n} = 1\,,\; a^b = a^{1+p^{m-1}} \rangle$, where $m \geqslant 2$ and $n \geqslant 1$;

(2) $G = \langle a, b \,|\, a^{p^m} = b^{p^n} = c^p = 1, [a,b] = c, [a,c] = [b,c] = 1 \rangle$, where $m + n > 2$ if $p = 2$;

(3) $G \simeq Q_8$.

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N. Blackburn – 1961

(1) G is a group of order p^3 and exponent p.

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(4) G has order 3^4 and $G = \langle c \rangle \big(\langle b \rangle \ltimes \langle a \rangle \big)$ where $a^9 = b^3 = 1$, $a^b = a^4 = ac^{-3}$, $b^c = b$ and $a^c = ba$.

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(5) $G = \langle c \rangle \big(\langle b \rangle \ltimes \langle a \rangle\big)$, where $a^4 = b^4 = 1$, $[a, b] = [b, c] = a^2$, $[a, c] = b^2 = c^2$; in particular, G has order 2^5, $d(G) = 3$ and $G' = \Phi(G) = Z(G) = \Omega_1(G)$.

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Let G be a non-periodic soluble metahamiltonian group. Then

(5) G' is finite and abelian.

Preliminaries on metahamiltonian groups

A group G is said to be *locally graded* if every finitely generated non-trivial subgroup of G contains a proper subgroup of finite index.

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Let G be a non-periodic soluble metahamiltonian group. Then

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Let G be a locally graded metahamiltonian group. Then

(6) G' is finite of prime power order and G'' is abelian.

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Nilpotency class 2 and cyclic commutator subgroup

N.F. Kuzennyĭ, N.N. Semko – 1987

Let p be a prime and let κ be an integer > 1. A group G is a metahamiltonian p-group of nilpotency class 2 with G' cyclic of order p^κ if and only if one of the following hold:

(1) $G = \big(\langle b\rangle\langle a\rangle\big) \times A$, where $[a, b] = a^{p^{\alpha-\kappa}}$, $o(a) = p^\alpha$, A is abelian of exponent $\leqslant p^{\alpha-2\kappa+1}$ and $o(bG') \geqslant \exp(A)$.

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(2) $G = \big(B \ltimes \langle a\rangle\big) \times A$, where $B = \langle b\rangle \times \langle b_1\rangle \times \ldots \times \langle b_t\rangle$, $o(a) = p^\alpha$, $p^{\alpha-\kappa} \geqslant o(b) = p^\beta \geqslant o(b_1) = p^{\beta_1} \geqslant \ldots \geqslant o(b_t) = p^{\beta_t}$, $[a,b] = a^{p^{\alpha-\kappa}}$, $[a,b_i] = a^{p^{\alpha-\kappa_i}}$ for $1 \leqslant i \leqslant t$, $0 < \kappa_t < \ldots < \kappa_1 < \min\{\kappa, \alpha-\kappa-\beta+2\}$, $\beta-\kappa > \beta_1-\kappa_1 > \ldots > \beta_t-\kappa_t$ and A is abelian of exponent $\leqslant p^{\alpha-2\kappa+1}$.

Nilpotency class 2 and cyclic commutator subgroup

N.F. Kuzennyĭ, N.N. Semko – [illegible]

Let p be a prime and let κ be an integer > 1. A group G is a metahamiltonian p-group of nilpotency class [illegible] with G' cyclic of order [illegible] if and only if one of the following held:

(1) $G = (\langle b\rangle\langle a\rangle) \times A$, where [illegible] $b] = a^{p^{\alpha-\kappa}}$, [illegible] $(a) =$ [illegible], A is abelian of exponent $\leqslant p^{[illegible]-2[illegible]+1}$ and [illegible] $(b$ [illegible] $') \geqslant \exp(A)$.

(2) $G = (B \ltimes \langle$ [illegible] $\times A$, where $B = \langle$ [illegible] $\times \langle b_1\rangle \times \ldots \times \langle b_t\rangle$, $o(a) = p^{\alpha}$, [illegible] $\geqslant o(b$ [illegible] $\geqslant o(b_1)$ [illegible] $= p^{\beta_1} \geqslant \ldots \geqslant o(b_t) = p^{\beta_t}$,
$[a, b]$ [illegible] $= a^{p^{\alpha-}}$ [illegible], [illegible] $_i]$ [illegible] $^{-\kappa_i}$ for $1 \leqslant i \leqslant t$,
$0 < \kappa_t <$ [illegible] $\ldots$ [illegible] κ_1 [illegible] $\min\{\kappa, \alpha - \kappa - \beta + 2\}$,
$\beta - \kappa$ [illegible] $\beta_1 -$ [illegible] $_1 > \ldots > \beta_t - \kappa_t$ and A is abelian of exponent $\leqslant p^{\alpha-}$ [illegible] $^{\kappa+1}$.

Nilpotency class 2 and cyclic commutator subgroup

N.F. Kuzennyĭ, N.N. Semko – 1987

Let p be a prime and let κ be an integer > 1. A group G is a metahamiltonian p-group of nilpotency class 2 with G' cyclic of order p^κ if and only if one of the following hold:

(1) $G = \big(\langle b\rangle\langle a\rangle\big) \times A$, where $[a,b] = a^{p^{\alpha-\kappa}}$, $o(a) = p^\alpha$, A is abelian of exponent $\leqslant p^{\alpha-2\kappa+1}$ and $o(bG') \geqslant \exp(A)$.

(2) $G = \big(B \ltimes \langle a\rangle\big) \times A$, where $B = \langle b\rangle \times \langle b_1\rangle \times \ldots \times \langle b_t\rangle$, $o(a) = p^\alpha$, $p^{\alpha-\kappa} \geqslant o(b) = p^\beta \geqslant o(b_1) = p^{\beta_1} \geqslant \ldots \geqslant o(b_t) = p^{\beta_t}$, $[a,b] = a^{p^{\alpha-\kappa}}$, $[a,b_i] = a^{p^{\alpha-\kappa_i}}$ for $1 \leqslant i \leqslant t$, $0 < \kappa_t < \ldots < \kappa_1 < \min\{\kappa, \alpha-\kappa-\beta+2\}$, $\beta-\kappa > \beta_1-\kappa_1 > \ldots > \beta_t-\kappa_t$ and A is abelian of exponent $\leqslant p^{\alpha-2\kappa+1}$.

Nilpotency class 2 and cyclic commutator subgroup

N.F. Kuzennyĭ, N.N. Semko... M.B., M. Ferrara, M. Trombetti – 1987-2021

Let p be a prime and let κ be an integer > 1. A group G is a metahamiltonian p-group of nilpotency class 2 with G' cyclic of order p^κ if and only if one of the following non-isomorphic conditions hold:

(1) $G = \big(\langle b\rangle\langle a\rangle\big) \times A$, where $[a,b] = a^{p^{\alpha-\kappa}}$, $o(a) = p^\alpha$, A is abelian of exponent $\leqslant p^{\alpha-2\kappa+1}$ and $o(bG') \geqslant \exp(A)$; distinct isomorphism types for A and $\langle a,b\rangle$ give rise to distinct isomorphism types for G.

(2) $G = \big(B \ltimes \langle a\rangle\big) \times A$, where $B = \langle b\rangle \times \langle b_1\rangle \times \ldots \times \langle b_t\rangle$, $o(a) = p^\alpha$, $p^{\alpha-\kappa} \geqslant o(b) = p^\beta \geqslant o(b_1) = p^{\beta_1} \geqslant \ldots \geqslant o(b_t) = p^{\beta_t}$, $[a,b] = a^{p^{\alpha-\kappa}}$, $[a,b_i] = a^{p^{\alpha-\kappa_i}}$ for $1 \leqslant i \leqslant t$, $0 < \kappa_t < \ldots < \kappa_1 < \min\{\kappa, \alpha-\kappa-\beta+2\}$, $\beta-\kappa > \beta_1-\kappa_1 > \ldots > \beta_t-\kappa_t$ and A is abelian of exponent $\leqslant p^{\alpha-2\kappa+1}$; distinct isomorphism types for A and distinct values of the parameters $\alpha, \beta, \beta_1, \ldots, \beta_t, \kappa, \kappa_1, \ldots, \kappa_t, t$ define distinct isomorphism types for G.

A roadmap

1) Minimal non-metacyclic p-groups.
2) Preliminaries on metahamiltonian groups.
3) Non-soluble metahamiltonian groups.
4) Soluble non-nilpotent metahamiltonian groups.
5) Preliminaries on nilpotent metahamiltonian groups.
6) Metahamiltonian p-groups of nilpotency class 2 whose commutator subgroup...
   - 6a) ... is cyclic.
   - 6b) ... is elementary abelian of rank 2.
   - 6c) ... is neither elementary abelian nor cyclic.
   - 6d) ... is elementary abelian of rank 3.
7) Metahamiltonian p-groups of nilpotency class 3.
8) Non-periodic nilpotent metahamiltonian groups.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko – 1998-1990

Let p be a prime and let G be a p-group of nilpotency class 2 where G' is elementary abelian of order p^2. Then G is metahamiltonian if and only if $G = H \times A$, where H and A satisfy the following conditions

(1) A is an abelian p-group with exponent smaller than that of any minimal non-abelian subgroup of H.

(2) If $H' = \Omega_1(H)$, then

- (2a) $H = \langle c\rangle(\langle b\rangle \ltimes \langle a\rangle)$, where $a^4 = b^4 = 1$, $[a,b] = [b,c] = a^2$, $[a,c] = b^2 = c^2$;
- (2b) $H=\langle d\rangle(\langle c\rangle(\langle b\rangle \ltimes \langle a\rangle))$, where $a^4 = b^4 = [a,d] = 1$, $[a,b] = [b,c] = [c,d] = a^2$, $[a,c] = b^2 = c^2 = d^2$, $[b,d] = a^2b^2$.

(3) If $H' < \Omega_1(H)$, then $H = \langle c\rangle \ltimes (\langle b\rangle \ltimes \langle a\rangle)$, where $o(a) = p^{\alpha} > p$, $o(b) = p^{\beta} > p$ and $o(c) = p^{\gamma} > 1$ and one of the following alternatives holds:

- (3a) $\gamma \geqslant \alpha = \beta + 1$, $[a,b] = 1$, $[a,c] = b^{p^{\beta-1}}$, $[b,c] = a^{sp^{\alpha-1}}$, $s = 1,\dots,p-1$;
- (3b) $\gamma \geqslant \alpha = \beta$, $[a,b] = 1$, $[a,c] = b^{p^{\beta-1}}$, $[b,c] = a^{sp^{\alpha-1}}b^{tp^{\beta-1}}$, $n^2 - tn - s \not\equiv_p 0$, $s,n = 1,\dots,p-1$, $t = 0,\dots,p-1$;
- (3c) $\gamma < \alpha = \beta$, $[a,b] = a^{p^{\alpha-1}}$, $[a,c] = b^{p^{\beta-1}}$, $[b,c] = a^{sp^{\alpha-1}}b^{tp^{\beta-1}}$, $n^2 - tn - s \not\equiv_p 0$, $s,n = 1,\dots,p-1$, $t = 0,\dots,p-1$;
- (3d) $\gamma < \alpha \geqslant \beta$, $p > 2$ when $\alpha = \beta = 2$, $[a,b] = a^{p^{\alpha-1}}$, $[a,c] = b^{p^{\beta-1}}$, $[b,c] = 1$;
- (3e) $\alpha = \beta + 1$, $\gamma < \alpha$, $[a,b] = a^{p^{\alpha-1}}$, $[a,c] = b^{p^{\beta-1}}$, $[b,c] = a^{sp^{\alpha-1}}$, $s = 1,\dots,p-1$.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko – 1998-1990

Let p be a prime and let G be a p-group of nilpotency class 2 where G' is [illegible]elian of ord[illegible] p^2 [illegible]hen [illegible] G is metahamiltonian if and only if $G = H \times A$, where H and A satisfy the fo[illegible]owing [illegible]ti[illegible]

(1) A is an abelian p-group with exponent smaller th[illegible] of any m[illegible]imal [illegible]n-[illegible] subgr[illegible]

(2) If $H' = \Omega_1(H)$, then

(2a) $H = \langle c\rangle(\langle b\rangle \ltimes \langle a\rangle)$, whe[illegible] $^4 = 1$, [illegible]$, b] = [b, c]$ [illegible] a^2, [illegible] $c]$ [illegible] b^2 [illegible] ^{2};

(2b) $H=\langle d\rangle(\langle c\rangle(\langle b\rangle \ltimes$ [illegible]$a\rangle))$ [illegible]here a [illegible] $b^4 =$ [illegible]$, d]=1, [a, b$[illegible] $= [b,$ [illegible] $= a^2$, $[a, c] = b^2 = c^2 =$ [illegible] $[b$[illegible]$a] = a^2 b^2$.

(3) If $H' < \Omega_1(H)$, t[illegible] $H =$ [illegible]$\rangle$ [illegible]$(\langle b\rangle \ltimes \langle a\rangle)$ where [illegible]$(a) = p^{\alpha} > p$, $o(b) = p^{\beta} > p$ and

o[illegible] $p^{\gamma} > 1$ an[illegible] of t[illegible] fo[illegible]owing alte[illegible]atives [illegible]lds:

(3a) [illegible] $\geqslant \alpha = \beta$ [illegible] 1, [illegible] $=$ [illegible] $[a, c]$ [illegible] $b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}}$, $s = 1, \ldots, p-1$;

(3b) γ [illegible] $\geqslant \alpha = \beta$ [illegible] $a, b]$ [illegible] $[a,$ [illegible] $^{\beta-1}$, $[b, c] = a^{sp^{\alpha-1}} b^{tp^{\beta-1}}$, $n^2 - tn - s \not\equiv_p 0$, s[illegible] $= 1, \ldots$ [illegible] -1, [illegible]$0, \ldots,$ [illegible] -1;

(3c) γ [illegible] $\alpha = \beta$, [illegible]$, b] =$ [illegible]$^{\alpha-1}$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}} b^{tp^{\beta-1}}$, n[illegible] $- tn - s$[illegible]$_p\ 0$, $s, n = 1, \ldots, p-1$, $t = 0, \ldots, p-1$;

(3d) [illegible] $< \alpha \geqslant \beta$, $p > 2$ when $\alpha = \beta = 2$, $[a, b] = a^{p^{\alpha-1}}$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = 1$;

(3e) [illegible] $= \beta + 1$, $\gamma < \alpha$, $[a, b] = a^{p^{\alpha-1}}$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}}$, $s = 1, \ldots, p-1$.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko – 1998-1990

Let p be a prime and let G be a p-group of nilpotency class 2 where G' is elementary abelian of order p^2. Then G is metahamiltonian if and only if $G = H \times A$, where H and A satisfy the following conditions

(1) A is an abelian p-group with exponent smaller than that of any minimal non-abelian subgroup of H.

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(2a) $H = \langle c\rangle\big(\langle b\rangle \ltimes \langle a\rangle\big)$, where $a^4 = b^4 = 1$, $[a, b] = [b, c] = a^2$, $[a, c] = b^2 = c^2$;

(2b) $H{=}\langle d\rangle\big(\langle c\rangle\big(\langle b\rangle \ltimes \langle a\rangle\big)\big)$, where $a^4{=}b^4{=}[a, d]{=}1$, $[a, b]{=}[b, c]{=}[c, d]{=}a^2$, $[a, c]{=}b^2{=}c^2{=}d^2$, $[b, d]{=}a^2b^2$.

(3) If $H' < \Omega_1(H)$, then $H = \langle c\rangle \ltimes \big(\langle b\rangle \ltimes \langle a\rangle\big)$, where $o(a) = p^{\alpha} > p$, $o(b) = p^{\beta} > p$ and $o(c) = p^{\gamma} > 1$ and one of the following alternatives holds:

(3a) $\gamma \geqslant \alpha = \beta + 1$, $[a, b] = 1$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}}$, $s = 1, \ldots, p-1$;

(3b) $\gamma \geqslant \alpha = \beta$, $[a, b] = 1$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}}b^{tp^{\beta-1}}$, $n^2 - tn - s \not\equiv_p 0$, $s, n = 1, \ldots, p-1$, $t = 0, \ldots, p-1$;

(3c) $\gamma < \alpha = \beta$, $[a, b] = a^{p^{\alpha-1}}$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}}b^{tp^{\beta-1}}$, $n^2 - tn - s \not\equiv_p 0$, $s, n = 1, \ldots, p-1$, $t = 0, \ldots, p-1$;

(3d) $\gamma < \alpha \geqslant \beta$, $p > 2$ when $\alpha = \beta = 2$, $[a, b] = a^{p^{\alpha-1}}$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = 1$;

(3e) $\alpha = \beta + 1$, $\gamma < \alpha$, $[a, b] = a^{p^{\alpha-1}}$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}}$, $s = 1, \ldots, p-1$.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko... M.B., M. Ferrara, M. Trombetti – 1998,1990-2021

Let p be a prime and let G be a p-group of nilpotency class 2 where G' is elementary abelian of order p^2. Then G is metahamiltonian if and only if $G = H \times A$, where H and A satisfy the following conditions

(1) A is an abelian p-group with exponent smaller than that of any minimal non-abelian subgroup of H.

(2) If $H' = \Omega_1(H)$, then

  (2a) $H = \langle c\rangle(\langle b\rangle \ltimes \langle a\rangle)$, where $a^4 = b^4 = 1$, $[a, b] = [b, c] = a^2$, $[a, c] = b^2 = c^2$;

  (2b) $H=\langle d\rangle(\langle c\rangle(\langle b\rangle \ltimes \langle a\rangle))$, where $a^4=b^4=[a, d]=1$, $[a, b]=[b, c]=[c, d]=a^2$, $[a, c]=b^2=c^2=d^2$, $[b, d]=a^2b^2$.

(3) If $H' < \Omega_1(H)$, then $H = \langle c\rangle \ltimes (\langle b\rangle \ltimes \langle a\rangle)$, where $o(a) = p^{\alpha} > p$, $o(b) = p^{\beta} > p$ and $o(c) = p^{\gamma} > 1$ and one of the following non-isomorphic alternatives holds:

  (3a) $[a, b] = 1$, $\alpha = \beta$, $[a, c] = b^{2^{\alpha-1}}$, $[b, c] = a^{2^{\alpha-1}} b^{2^{\alpha-1}}$; moreover, if $\alpha = 2$, then $\gamma = 1$;

  (3b) $[a, b] = 1$, $p > 2$, $\alpha = \beta$, $[a, c] = b^{p^{\alpha-1}}$, $[b, c] = a^{\nu p^{\alpha-1}}$ and ν is the smallest positive integer which is a quadratic non-residue modulo p;

  $(3c)_{\delta}$ $[a, b] = 1$, $p > 2$, $\alpha = \beta$, $[a, c] = b^{p^{\alpha-1}}$ and $[b, c] = a^{\delta p^{\alpha-1}} b^{p^{\alpha-1}}$, for $1 \leqslant \delta \leqslant p-1$ such that $1 + 4\delta$ is a quadratic non-residue modulo p;

  (3d) $[a, b] = 1$, $\alpha = \beta + 1$, $[a, c] = b^{p^{\beta-1}}$ and $[b, c] = a^{p^{\alpha-1}}$;

  (3e) $[a, b] = 1$, $\alpha = \beta + 1$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{\nu p^{\alpha-1}}$ and ν is the smallest positive integer which is a quadratic non-residue modulo p;

  (3f) $[a, b] = a^{p^{\alpha-1}}$, $\gamma < \alpha$, $\gamma \leqslant \beta \leqslant \alpha$, $[a, c] = b^{p^{\beta-1}}$ and $[b, c] = 1$; moreover, if $\alpha = \beta = 2$, then $p > 2$.

Distinct isomorphism classes for A and H give rise to distinct isomorphism classes for G.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko... M.B., M. Ferrara, M. Trombett[illegible]98,19[illegible]2021[illegible]

Let p be a prime and let G be a p-group of nilpotency class 2 where G' is el[illegible]ry [illegible]belian of ord[illegible] p[illegible] Then [illegible] G is metahamiltonian if and only if $G = H \times A$, where H and A satisfy the f[illegible]ing [illegible]ons

(1) A is an abelian p-group with exponent smaller than th[illegible] of any m[illegible]imal n[illegible]ubgr[illegible]p[illegible]

(2) If $H' = \Omega_1(H)$, then

(2a) $H = \langle c\rangle(\langle b\rangle \ltimes \langle a\rangle)$, where [illegible] $b^4 = 1$, [illegible]$, b] = [b, c]$ [illegible]$a, c]$ [illegible] $b^2 = c^2$[illegible]

(2b) $H = \langle d\rangle(\langle c\rangle(\langle b\rangle \ltimes \langle a\rangle))$, [illegible] $= b^4 =$ [illegible]$, d] = 1$, $[a, b$[illegible] $= [b$[illegible] $=$ [illegible] d[illegible],
$[a, c] = b^2 = c^2 =$ [illegible], $[b$[illegible] $= a^2 b$[illegible]

(3) If $H' < \Omega_1(H)$, t[illegible]n $H =$ [illegible]$\rangle$ [illegible]$(\langle b\rangle \ltimes \langle a\rangle)$ where [illegible]$(a) = p^{\alpha} >$ [illegible], $o(b) = p^{\beta} > p$ and $o(c) = p^{\gamma} > 1$ ar[illegible]e of t[illegible] f[illegible]lowing non[illegible]somor[illegible]ic alternatives holds:

[illegible] $[a, b] = 1$, [illegible], $[$[illegible]$c]$ [illegible] $b^{2^{\alpha-1}}$ [illegible]$b, c] =$ [illegible]$^{2^{\alpha-1}} b^{2^{\alpha-1}}$; moreover, if $\alpha = 2$, then $\gamma = 1$;

(3b) [illegible]$, b] = 1$, [illegible] > 2 [illegible]β[illegible]$c] =$ [illegible]$^{\alpha-1}$, $[b, c] = a^{\nu p^{\alpha-1}}$ and ν is the smallest positive i[illegible]eger whic[illegible] is a [illegible]ati[illegible]due modulo p;

(3c)$_\delta$ $[$[illegible]$b] = 1$, p [illegible] > 2, α [illegible]β, $[a, c] = b^{p^{\alpha-1}}$ and $[b, c] = a^{\delta p^{\alpha-1}} b^{p^{\alpha-1}}$, for $1 \leqslant \delta \leqslant p-1$ su[illegible]h that $1 +$ [illegible]δ is a q[illegible]dratic non-residue modulo p;

(3d) $[$[illegible]$b] = 1$, α [illegible] $\beta + 1$, $[a, c] = b^{p^{\beta-1}}$ and $[b, c] = a^{p^{\alpha-1}}$;

(3e) [illegible]$b] = 1$, $\alpha = \beta + 1$, $[a, c] = b^{p^{\beta-1}}$, $[b, c] = a^{\nu p^{\alpha-1}}$ and ν is the smallest positive integer [illegible]hich is a quadratic non-residue modulo p;

[illegible] $[a, b] = a^{p^{\alpha-1}}$, $\gamma < \alpha$, $\gamma \leqslant \beta \leqslant \alpha$, $[a, c] = b^{p^{\beta-1}}$ and $[b, c] = 1$; moreover, if $\alpha = \beta = 2$, then $p > 2$.

Distinct isomorphism classes for A and H give rise to distinct isomorphism classes for G.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko... M.B., M. Ferrara, M. Trombetti – 1998,1990-2021

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- (2b) $H = \langle d\rangle(\langle c\rangle(\langle b\rangle \ltimes \langle a\rangle))$, where $a^4 = b^4 = [a,d] = 1$, $[a,b] = [b,c] = [c,d] = a^2$, $[a,c] = b^2 = c^2 = d^2$, $[b,d] = a^2b^2$.

(3) If $H' < \Omega_1(H)$, then $H = \langle c\rangle \ltimes (\langle b\rangle \ltimes \langle a\rangle)$, where $o(a) = p^\alpha > p$, $o(b) = p^\beta > p$ and $o(c) = p^\gamma > 1$ and one of the following non-isomorphic alternatives holds:

- (3a) $[a,b] = 1$, $\alpha = \beta$, $[a,c] = b^{2^{\alpha-1}}$, $[b,c] = a^{2^{\alpha-1}}b^{2^{\alpha-1}}$; moreover, if $\alpha = 2$, then $\gamma = 1$;
- (3b) $[a,b] = 1$, $p > 2$, $\alpha = \beta$, $[a,c] = b^{p^{\alpha-1}}$, $[b,c] = a^{\nu p^{\alpha-1}}$ and ν is the smallest positive integer which is a quadratic non-residue modulo p;
- (3c)$_\delta$ $[a,b] = 1$, $p > 2$, $\alpha = \beta$, $[a,c] = b^{p^{\alpha-1}}$ and $[b,c] = a^{\delta p^{\alpha-1}}b^{p^{\alpha-1}}$, for $1 \leqslant \delta \leqslant p-1$ such that $1+4\delta$ is a quadratic non-residue modulo p;
- (3d) $[a,b] = 1$, $\alpha = \beta+1$, $[a,c] = b^{p^{\beta-1}}$ and $[b,c] = a^{p^{\alpha-1}}$;
- (3e) $[a,b] = 1$, $\alpha = \beta+1$, $[a,c] = b^{p^{\beta-1}}$, $[b,c] = a^{\nu p^{\alpha-1}}$ and ν is the smallest positive integer which is a quadratic non-residue modulo p;
- (3f) $[a,b] = a^{p^{\alpha-1}}$, $\gamma < \alpha$, $\gamma \leqslant \beta \leqslant \alpha$, $[a,c] = b^{p^{\beta-1}}$ and $[b,c] = 1$; moreover, if $\alpha = \beta = 2$, then $p > 2$.

Distinct isomorphism classes for A and H give rise to distinct isomorphism classes for G.

Metahamiltonian groups - a problem

Sufficient conditions for theorems on metahamiltonian groups are

not obvious

although someone states the contrary.

A roadmap

1) Minimal non-metacyclic p-groups.
2) Preliminaries on metahamiltonian groups.
3) Non-soluble metahamiltonian groups.
4) Soluble non-nilpotent metahamiltonian groups.
5) Preliminaries on nilpotent metahamiltonian groups.
6) Metahamiltonian p-groups of nilpotency class 2 whose commutator subgroup...
   - 6a) ... is cyclic.
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1) Minimal non-metacyclic p-groups.
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Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko – 1989

A group G is a locally finite metahamiltonian p-group with nilpotency class 2 with G' of rank 3 if and only if $G = H \times C$, where C is abelian and $\exp(C) \leqslant p^{\gamma-1}$ and H satisfies the following conditions.

(1) $H = \big(\langle a\rangle \ltimes \big(\langle b\rangle \times \langle c^{p^{\gamma-1}}\rangle\big)\big)\langle c\rangle$, where

- $p^{\alpha} = o(a) \geqslant p^{\beta} = o(b) \geqslant p^{\gamma} = o(c) \geqslant p^2$,
- $Z(H) = \Phi(H) = \langle a^p\rangle \times \langle b^p\rangle \times \langle c^p\rangle$,
- $H' = \langle a^{p^{\alpha-1}}\rangle \times \langle b^{p^{\beta-1}}\rangle \times \langle c^{p^{\gamma-1}}\rangle = \Omega_1(H)$ and $[b,a] = c^{p^{\gamma-1}}$.

(2) And one of the following alternatives holds:

(2a) $[a,c] = b^{kp^{\beta-1}}$, $[b,c] = a^{sp^{\alpha-1}}b^{tp^{\beta-1}}$, $\alpha-1=\beta-1=\gamma$, $n^2 - tn - ks \not\equiv_p 0$, $k,s,n = 1,\dots,p-1$, $t = 0,\dots,p-1$;

(2b) $[a,c] = b^{kp^{\beta-1}}c^{\Delta p^{\gamma-1}}$, $[b,c] = a^{sp^{\alpha-1}}$, $\alpha-1=\beta=\gamma$, $n^2 - \Delta n - k \not\equiv_p 0$, $k,s,n = 1,\dots,p-1$, $\Delta = 0,\dots,p-1$;

(2c) $p=\alpha=\beta=\gamma=2$, $[a,c] = b^2c^2$, $[b,c] = a^2b^2$;

(2d) $p=2$, $\alpha=\beta=\gamma>2$, $[a,c] = b^{p^{\beta-1}}c^{p^{\gamma-1}}$, $[b,c] = a^{p^{\alpha-1}}b^{p^{\beta-1}}c^{p^{\gamma-1}}$;

(2e) $p>2$, $\alpha=\beta=\gamma$, $[a,c] = b^{kp^{\beta-1}}c^{\Delta p^{\gamma-1}}$, $[b,c] = a^{sp^{\alpha-1}}b^{tp^{\beta-1}}c^{lp^{\gamma-1}}$, $n_1^2 - \Delta n_1 - k \not\equiv_p 0$, $n_2^2 - tn_2 - ks \not\equiv_p 0$, $n_3^2 - ln_3 - s \not\equiv_p 0$, $k,s,n_1,n_2,n_3 = 1,\dots,p-1$, $\Delta,t,l = 0,\dots,p-1$.

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko – 1989

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- $p^{\alpha} = o(a) \geqslant p^{\beta} = o(b) \geqslant p^{\gamma} = o(c) \geqslant p^2$,
- $Z(H) = \Phi(H) = \langle a^p \rangle \times \langle b^p \rangle \times \langle c^p \rangle$,
- $H' = \langle a^{p^{\alpha-1}} \rangle \times \langle b^{p^{\beta-1}} \rangle \times \langle c^{p^{\gamma-1}} \rangle = \Omega_1(H)$ and $[b, a] = c^{p^{\gamma-1}}$.

(2) And one of the following alternatives holds:

(2a) $[a, c] = b^{kp^{\beta-1}}$, $[b, c] = a^{sp^{\alpha-1}} b^{tp^{\beta-1}}$, $\alpha - 1 = \beta - 1 = \gamma$, $n^2 - tn - ks \not\equiv_p 0$, $k, s, n = 1, \dots, p-1$, $t = 0, \dots, p-1$;

(2b) $[a, c] = b^{kp^{\beta-1}} c^{\Delta p^{\gamma-1}}$, $[b, c] = a^{sp^{\alpha-1}}$, $\alpha - 1 = \beta = \gamma$, $n^2 - \Delta n - k \not\equiv_p 0$, $k, s, n = 1, \dots, p-1$, $\Delta = 0, \dots, p-1$;

(2c) $p = \alpha = \beta = \gamma = 2$, $[a, c] = b^2 c^2$, $[b, c] = a^2 b^2$;

~~(2d) $p = 2$, $\alpha = \beta = \gamma > 2$, $[a, c] = b^{p^{\beta-1}} c^{p^{\gamma-1}}$, $[b, c] = a^{p^{\alpha-1}} b^{p^{\beta-1}} c^{p^{\gamma-1}}$;~~

~~(2e) $p > 2$, $\alpha = \beta = \gamma$, $[a, c] = b^{kp^{\beta-1}} c^{\Delta p^{\gamma-1}}$, $[b, c] = a^{sp^{\alpha-1}} b^{tp^{\beta-1}} c^{lp^{\gamma-1}}$, $n_1^2 - \Delta n_1 - k \not\equiv_p 0$, $n_2^2 - tn_2 - ks \not\equiv_p 0$, $n_3^2 - ln_3 - s \not\equiv_p 0$, $k, s, n_1, n_2, n_3 = 1, \dots, p-1$, $\Delta, t, l = 0, \dots, p-1$.~~

Nilpotency class 2 and $G' \simeq \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$

N.F. Kuzennyĭ, N.N. Semko... M.B., M. Ferrara, M. Trombetti – 1989-2021

Let p be a prime and let G be a locally finite metahamiltonian p-group of nilpotency class 2 with G' of rank 3. Then $G = H \times C$, where H and C satisfy the following conditions.

(1) $H = \big(\langle a\rangle \ltimes (\langle b\rangle \times \langle c^{p^{\gamma-1}}\rangle)\big)\langle c\rangle$, where

- (1a) $p^\alpha = o(a) \geqslant p^\beta = o(b) \geqslant p^\gamma = o(c) \geqslant p^2$,
- (1b) $Z(H) = \Phi(H) = \langle a^p\rangle \times \langle b^p\rangle \times \langle c^p\rangle$,
- (1c) $H' = \langle a^{p^{\alpha-1}}\rangle \times \langle b^{p^{\beta-1}}\rangle \times \langle c^{p^{\gamma-1}}\rangle = \Omega_1(H)$,
- (1d) $[b,a] = c^{p^{\gamma-1}}$ and either $\beta = \gamma$ or $\beta = \gamma + 1$.

(2) If $\beta = \gamma + 1$, then $\alpha = \beta$ and one of the following non-isomorphic alternatives hold:

- (2a) $p = 2$, $[a,c] = b^{p^{\alpha-1}}$ and $[c,b] = a^{p^{\alpha-1}} b^{p^{\alpha-1}}$;
- (2b) $p \equiv_4 3$, $[a,c] = b^{p^{\alpha-1}}$ and $[c,b] = a^{p^{\alpha-1}}$;
- (2c) $p \equiv_4 1$, $[a,c] = b^{\varepsilon p^{\alpha-1}}$ and $[c,b] = a^{p^{\alpha-1}}$ ($0 < \varepsilon < p$ is non-square modulo p);
- $(2d)_\delta$ $p > 2$, $0 < \delta < p$, $[a,c] = b^{\delta p^{\alpha-1}}$, $[c,b] = a^{p^{\alpha-1}} b^{-\delta p^{\alpha-1}}$ and $\delta^2 - 4\delta$ is a quadratic non-residue modulo p.

(3) If $\beta = \gamma$, then either $\alpha = \beta$ or $\alpha = \beta + 1$, and one of the following non-isomorphic alternatives hold:

- (3a) If $\alpha = \beta$, then $\alpha = p = 2$, $[a,c] = b^2c^2$ and $[c,b] = a^2b^2$;
- If $\alpha = \beta + 1$, then:
- (3b) $p = 2$, then $[a,c] = b^{p^{\beta-1}} c^{p^{\gamma-1}}$ and $[c,b] = a^{p^{\alpha-1}}$;
- (3c) $p \equiv_4 3$, $[a,c] = b^{p^{\beta-1}}$ and $[c,b] = a^{p^{\alpha-1}}$;
- (3d) $p \equiv_4 1$, $[a,c] = b^{\varepsilon p^{\beta-1}}$ and $[c,b] = a^{\varepsilon p^{\alpha-1}}$ ($0 < \varepsilon < p$ is non-square modulo p);
- $(3e)_\delta$ $p > 2$, $0 < \delta < p$, $[a,c] = b^{\delta p^{\beta-1}} c^{\delta p^{\gamma-1}}$, $[c,b] = a^{\delta p^{\alpha-1}}$ and $\delta^2 - 4\delta$ is a quadratic non-residue modulo p.

(4) C is abelian and $\exp(C) \leqslant p^{\gamma-1}$.

+ Converse and iso.

A roadmap

1) Minimal non-metacyclic p-groups.
2) Preliminaries on metahamiltonian groups.
3) Non-soluble metahamiltonian groups.
4) Soluble non-nilpotent metahamiltonian groups.
5) Preliminaries on nilpotent metahamiltonian groups.
6) Metahamiltonian p-groups of nilpotency class 2 whose commutator subgroup...
   - 6a) ... is cyclic.
   - 6b) ... is elementary abelian of rank 2.
   - 6c) ... is neither elementary abelian nor cyclic.
   - 6d) ... is elementary abelian of rank 3.
7) Metahamiltonian p-groups of nilpotency class 3.
8) Non-periodic nilpotent metahamiltonian groups.

Nilpotency class 3 (p odd)

A.A. Mahnev – 1976

Т е о р е м а 3. При $p > 2$ следующие группы, и только они являются метагамильтоновыми p-группами класса больше 2:

(а) G – метациклическая группа, $\Phi(G') = Z(G) \cap G'$.

(б) $G' = \langle \pi \rangle \times \langle \tau \rangle \times \langle \sigma \rangle$, $\pi \in Z(G)$,

$$G = ((\langle x \rangle \times \langle \tau \rangle) \lambda \langle \sigma \rangle) \langle y \rangle, \quad x^p = \pi \tau^{\alpha}, \quad y^p = \pi^{\beta} \tau,$$

$$[x, y] = \sigma, \quad [x, \sigma] = \tau, \quad [y, \sigma] = \pi.$$

Nilpotency class 3 (p odd)

M.B., M. Ferrara, M. Trombetti – 2021

Let p be an odd prime and let G be a p-group of nilpotency class 3. Then G is metahamiltonian if and only if it satisfies one of the following conditions:

...

(2) $G = \langle g\rangle\big(\langle x\rangle \ltimes (\langle y\rangle \times \langle z\rangle)\big)$ where $[y, g] = x$, $[x, y] = z$, $o(y) = o(g) = p^2$, $o(x) = o(z) = p$, $|G| = p^5$, $Z(G) = \langle z, [x, g]\rangle$, $G' = \langle [x, g]\rangle \times \langle z\rangle \times \langle x\rangle$ is elementary abelian of order p^3 and one of the following non-isomorphic alternatives holds:

- (2a) $p = 3$, $g^3 = z$, $[x, g] = y^3$;
- (2b) $[x, g] = y^p$, $g^p = z^\nu$, where ν is the smallest positive integer which is a quadratic non-residue mod p;
- $(2c)_\delta$ $p > 3$, $[x, g] = y^{\delta p}$, $g^p = y^p z$, where $1 \leqslant \delta < p$ and $1 + 4\delta$ is a quadratic non-residue mod p.

...

Beware!

Thank you for your attention