Hausdorff dimension in profinite groups

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June 2023

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The filtration $\mathcal{S}$ induces a translation-invariant metric $d^{\mathcal{S}}$ on G defined as

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This metric, in turns, defines the *Hausdorff dimension function* $\text{hdim}_G^{\mathcal{S}}(X)$ for any subset $X \subseteq G$ with respect to the filtration series $\mathcal{S}$.

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Theorem (Y. Barnea, A. Shalev)

Let G be a countably based profinite group and let $\mathcal{S} : G = G_0 \geq G_1 \geq G_2 \geq \cdots$ be a filtration of G. If H is a closed subgroup of G, then

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \liminf_{n\to\infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|} \in [0, 1].$$

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Examples

Let G be a countably based profinite group and $\mathcal{S}$ any filtration series of G. Then:

- $\operatorname{hdim}_G^{\mathcal{S}}(N) = 1$ *for every open normal subgroup N of G.*
- $\operatorname{hdim}_G^{\mathcal{S}}(H) = 0$ *for every finite subgroup H of G.*

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Remark

By the examples above, $\{0,1\} \subseteq \mathrm{hspec}^{\mathcal{S}}(G)$ *for every filtration* $\mathcal{S}$ *of* G*.*

Filtration series

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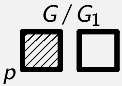

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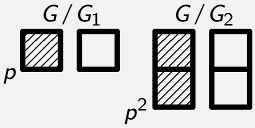

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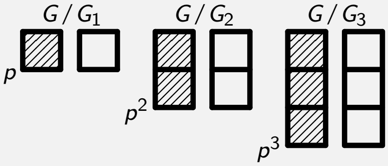

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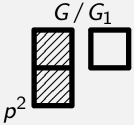

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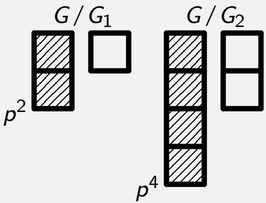

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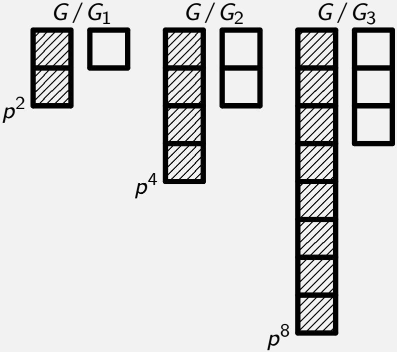

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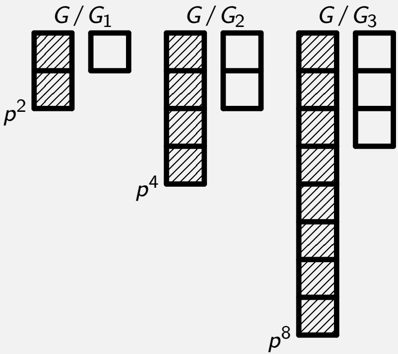

Similarly, if $\mathcal{S} : G = G_0 \geq G_1 \geq \cdots$ *such that* $G_i = \langle (p^i, 0), (0, p^{2^i}) \rangle$ *for every* $i \geq 0$, *then* $\mathrm{hdim}_G^{\mathcal{S}}(H) = 0$, *and* $\mathrm{hspec}^{\mathcal{S}}(G) = \{0, 1\}$.

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Theorem (B. Klopsch, A. Thillaisundaram, A. Zugadi-Reizabal)

Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$, *and let* $\mathcal{S} : G = G_0 \geq G_1 \geq \cdots$ *be a filtration of* G *such that*

$$G_i = \langle (p^{a_i}, ip^{a_i}), (0, p^{b_i}) \rangle,$$

where $a_0 = b_0 = 0$ *and* $a_i = p^i, b_i = p^i + 1$ *for* $i \geq 1$. *Then,*

$$\left[{}^{1}/_{p+1}\, , {}^{p-1}/_{p+1} \right] \subseteq \mathrm{hspec}^{\mathcal{S}}(G).$$

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Theorem (IH, B. Klopsch, A. Thillaisundaram)

Let $G = \mathbb{Z}_p \oplus \mathbb{Z}_p$, and let $I \subseteq [0,1]$ be finite with $0, 1 \in I$. Then there exists a filtration $\mathcal{S}$ of G such that

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We need to restrict our attention to specific filtration series.

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Standard filtration series for finitely generated pro-p groups:

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Finite Hausdorff spectra and p-adic analytic groups

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Problem 1

Let G be a finitely generated pro-p group and let $\mathcal{S} \in \{\mathcal{L}, \mathcal{D}, \mathcal{P}, \mathcal{I}, \mathcal{F}\}$. If G is p-adic analytic, does it follow that $\mathrm{hspec}^{\mathcal{S}}(G)$ *is finite?*

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Let G be a p-adic analytic pro-p group and H a closed subgroup of G. Then, for $\mathcal{S} \in \{\mathcal{D}, \mathcal{P}, \mathcal{I}, \mathcal{F}\}$, we have

$$\mathrm{hdim}_G^{\mathcal{S}}(H) = \frac{\dim(H)}{\dim(G)},$$

where $\dim(H)$ and $\dim(G)$ stand for the analytic dimension of H and G respectively.

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where $\dim(H)$ *and* $\dim(G)$ *stand for the analytic dimension of H and G respectively.*

In particular, if G is a p-adic analytic pro-p group, then

$$\mathrm{hspec}^{\mathcal{S}}(G) \subseteq \left\{0, \frac{1}{\dim(G)}, \ldots, \frac{\dim(G)-1}{\dim(G)}, 1\right\}$$

for any $\mathcal{S} \in \{\mathcal{D}, \mathcal{P}, \mathcal{I}, \mathcal{F}\}$.

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There exists a family of p-adic analytic pro-p groups $G(m, d)$, where $m, d \geq 0$, such that

$$\frac{|\,\mathrm{hspec}^{\mathcal{L}}(G(m, d))|}{\dim(G(m, d))} \to d + 1 \quad \text{as} \quad m \to \infty,$$

which is unbounded as d tends to infinity.

Complete answer to Problem 1

Theorem (IH, B. Klopsch, A. Thillaisundaram)

Let G be a p-adic analytic pro-p group of dimension d. Then there exist $\zeta_1, \ldots, \zeta_d \in [0,1] \cap \mathbb{Q}$ such that

$$\mathrm{hspec}^{\mathcal{L}} \subseteq \left\{ \frac{\epsilon_1\zeta_1 + \cdots + \epsilon_d\zeta_d}{\zeta_1 + \cdots + \zeta_d} \mid \epsilon_i \in \{0,1\} \right\}$$

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In particular, $|\,\mathrm{hspec}^{\mathcal{L}}(G)| \leq 2^d$. (This bound can be improved)

Explicit description of $\mathcal{L}$ for p-adic analytic groups

Definition

Let L be a $\mathbb{Z}_p$-module. Two filtrations $\mathcal{S} : L = L_0 \geq L_1 \geq \cdots$ and $\mathcal{S}^* : L = L_0^* \geq L_1^* \geq \cdots$ are said to be *equivalent* if there exists $b \in \mathbb{N}$ such that $|L_i + L_i^* : L_i \cap L_i^*| \leq b$ for every $i \in \mathbb{N}$.

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Definition

Let L be a free $\mathbb{Z}_p$-module of dimension d, and let $\{x_1, \ldots, x_d\}$ be a $\mathbb{Z}_p$-basis for L. We say that a filtration $\mathcal{S} : L = L_0 \geq L_1 \geq \cdots$ is a *split filtration* with respect to $\{x_1, \ldots, x_d\}$ with growth rates $\zeta_1, \ldots, \zeta_d \in (0, 1]$ if it is equivalent to the filtration $\mathcal{S}^* : L = L_0^* \geq L_1^* \geq \cdots$, where

$$L_i^* = \langle p^{\lfloor i\zeta_1 \rfloor} x_1, p^{\lfloor i\zeta_2 \rfloor} x_2, \ldots, p^{\lfloor i\zeta_d \rfloor} x_d \rangle_{\mathbb{Z}_p}$$

for every $i \in \mathbb{N}_0$.

Explicit description of $\mathcal{L}$ for p-adic analytic groups

What does $L_i^* = \langle p^{\lfloor i\zeta_1 \rfloor} x_1, p^{\lfloor i\zeta_2 \rfloor} x_2, \ldots, p^{\lfloor i\zeta_d \rfloor} x_d \rangle_{\mathbb{Z}_p}$ mean?

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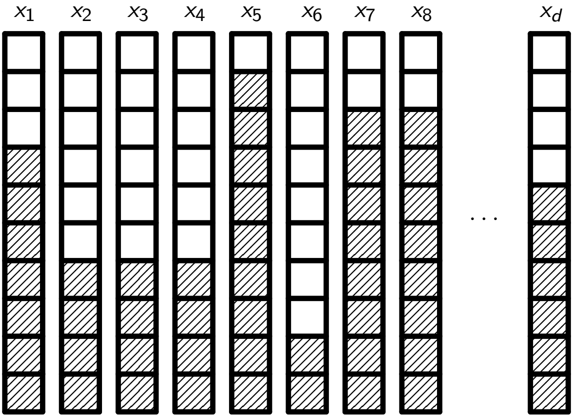

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Definition

Let G be a group acting on a $\mathbb{Z}_p$-module L. The *lower p-series of L with respect to G*, is the series

$$\mathcal{L}_L^G \colon P_0^G(L) = L, \quad \text{and} \quad P_i^G(L) = pP_{i-1}^G(L) + [P_{i-1}^G(L), G] \quad \text{for } i \geq 1.$$

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Theorem (IH, B. Klopsch, A. Thillaisundaram)

Let G be a p-adic analytic group acting on a free $\mathbb{Z}_p$-Lie algebra L. Then the series $\mathcal{L}_L^G$ is equivalent to a split filtration of L.

Explicit description of $\mathcal{L}$ for p-adic analytic groups

Corollary (IH, B. Klopsch, A. Thillaisundaram)

Let G be a p-adic analytic pro-p group of dimension d and let U be a uniform pro-p subgroup of G of finite index. Then there exist $b, c \in \mathbb{N}_0$, $x_1, \ldots, x_d \in U$ and $\zeta_1, \ldots, \zeta_d \in (0,1] \cap \mathbb{Q}$ such that

$$\langle x_1^{p^{\lfloor i\zeta_1 \rfloor + b}} \rangle \cdots \langle x_d^{p^{\lfloor i\zeta_d \rfloor + b}} \rangle \leq P_{c+i}(G) \leq \langle x_1^{p^{\lfloor i\zeta_1 \rfloor - b}} \rangle \cdots \langle x_d^{p^{\lfloor i\zeta_d \rfloor - b}} \rangle,$$

for every $i \in \mathbb{N}_0$, with

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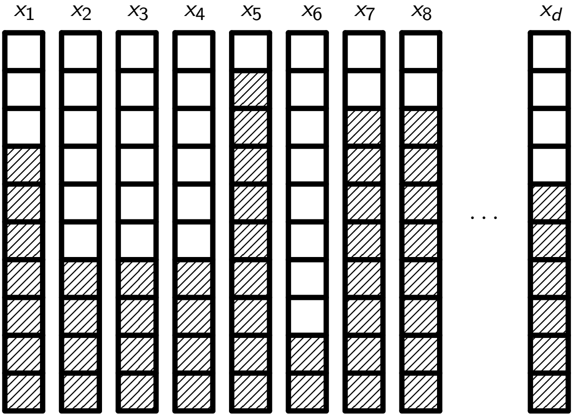
Explicit description of $\mathcal{L}$ for p-adic analytic groups

$P_{c+i}(G)$ is similar to $\langle x_1^{p^{\lfloor i\zeta_1 \rfloor}} \rangle \cdots \langle x_d^{p^{\lfloor i\zeta_d \rfloor}} \rangle$.



Finite Hausdorff spectra and *p*-adic analytic groups

Problem 1

Let G be a finitely generated pro-p group and let $\mathcal{S} \in \{\mathcal{L}, \mathcal{D}, \mathcal{P}, \mathcal{I}, \mathcal{F}\}$. If G is p-adic analytic, does it follow that $\mathrm{hspec}^{\mathcal{S}}(G)$ is finite?

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Partial answer to Problem 2

There exists a partial solution to this problem.

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Theorem (B. Klopsch, A. Thillaisundaram, A. Zugadi-Reizabal)

Let G be a finitely generated solvable pro-p group, and let $\mathcal{S} \in \{\mathcal{D}, \mathcal{P}, \mathcal{F}\}$. If G is not p-adic analytic, then the Hausdorff spectrum $\mathrm{hspec}^{\mathcal{S}}(G)$ with respect to $\mathcal{S}$ contains an infinite real interval.

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Corollary

Let G be a finitely generated solvable pro-p group and let $\mathcal{S} \in \{\mathcal{D}, \mathcal{P}, \mathcal{F}\}$. Then, G is p-adic analytic if and only if $\mathrm{hspec}^{\mathcal{S}}(G)$ is finite.

Finite Hausdorff spectra and p-adic analytic group

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4. *The group G is finite, or there exists a closed subgroup $H \leq G$ such that $H \cong \mathbb{Z}_p$ and $\mathrm{hdim}_G^{\mathcal{S}}(H) > 0$.*

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Most of the spectra of these groups cover the full interval $[0, 1]$.

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Definition

Let G be a countably based profinite group and let $\mathcal{S}$ be a filtration of G. The *normal Hausdorff spectrum* of G with respect to the filtration $\mathcal{S}$ is

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Theorem (B. Klopsch, A. Thillaisundaram)

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Does there exist a finitely generated pro-p group with full normal Hausdorff spectra?

Full normal Hausdorff spectra

Theorem (I.H., B. Klopsch)

There exists a 2-generator pro-p group $\mathfrak{G}(p)$, with p odd, such that

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Let $\mathcal{S}|_Z : Z = Z_0 \geq Z_1 \geq \cdots$, where $Z_i = Z \cap G_i$ for $i \in \mathbb{N}_0$, denote the induced filtration series of Z.

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Let $\mathcal{S}|_Z : Z = Z_0 \geq Z_1 \geq \cdots$, where $Z_i = Z \cap G_i$ for $i \in \mathbb{N}_0$, denote the induced filtration series of Z. Then for every $K \leq Z$,

$$\mathrm{hdim}_G^{\mathcal{S}}(K) = \mathrm{hdim}_G^{\mathcal{S}}(Z) \cdot \mathrm{hdim}_Z^{\mathcal{S}|_Z}(K).$$

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$$\begin{aligned} W &= C_p \,\hat{\wr}\, \mathbb{Z}_p \\ &= \langle \dot{x}, \dot{y} \rangle \end{aligned}$$

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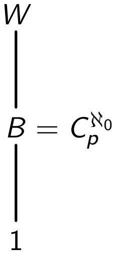

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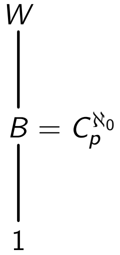
$$W = C_p \hat{\wr} \mathbb{Z}_p = \langle \dot{x}, \dot{y} \rangle$$

$F = \langle \tilde{x}, \tilde{y} \rangle$
free pro-p group
on 2 generators



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$$\varphi : F \longrightarrow W$$
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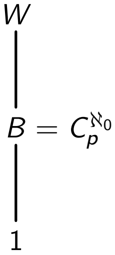

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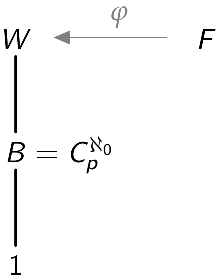

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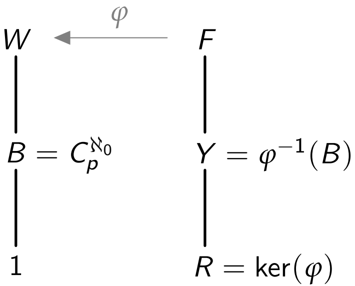

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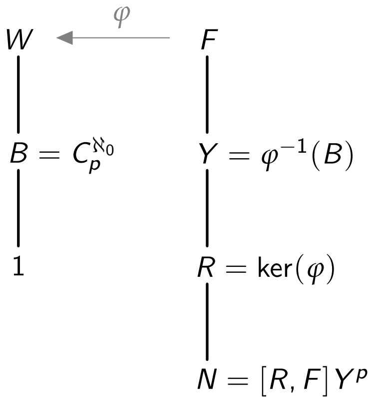

Construction of a finitely generated pro-p group with full normal Hausdorff spectra

$$W = C_p \,\hat{\wr}\, \mathbb{Z}_p = \langle \dot{x}, \dot{y} \rangle$$

$F = \langle \tilde{x}, \tilde{y} \rangle$
free pro-p group
on 2 generators

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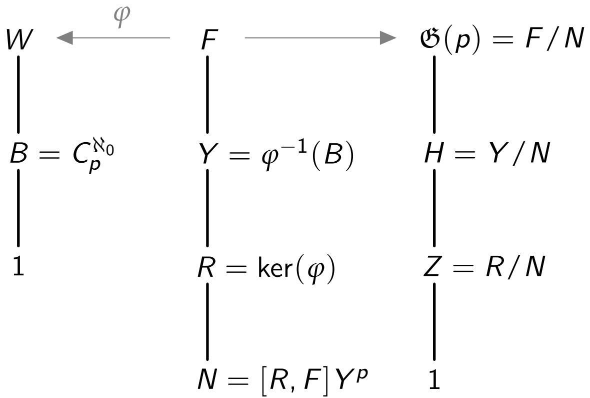

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$$W \xleftarrow{\ \varphi\ } F \longrightarrow \mathfrak{G}(p) = F/N$$

$$\begin{array}{ccc} W & F & \mathfrak{G}(p) = F/N \\ | & | & | \\ B = C_p^{\aleph_0} & Y = \varphi^{-1}(B) & H = Y/N \\ | & | & | \\ 1 & R = \ker(\varphi) & Z = R/N \\ & | & | \\ & N = [R, F]\,Y^p & 1 \end{array}$$

Lemma

In the group $\mathfrak{G}(p)$, the subgroup Z has strong Hausdorff dimension < 1.

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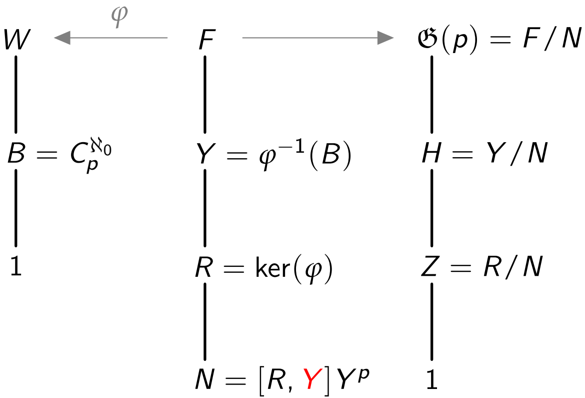

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Finitely generated Hausdorff spectrum

Finitely generated Hausdorff spectrum

Definition

Let G be a countably based profinite group and let $\mathcal{S}$ be a filtration of G. The *finitely generated Hausdorff spectrum* of G with respect to the filtration $\mathcal{S}$ is

$$\mathrm{hspec}^{\mathcal{S}}_{fg}(G) = \{\mathrm{hdim}^{\mathcal{S}}_{G}(H) \mid H \leq_c G \text{ and } H \text{ is finitely generated}\} \subseteq [0,1].$$

Finitely generated Hausdorff spectrum

Does there exist a pro-p group with infinite finitely generated Hausdorff spectrum?

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Theorem (M. Abért, B. Virág)

Let $G = \mathrm{Aut}(\mathcal{T})$ *be the group of automorphisms of the p-adic rooted tree* $\mathcal{T}$. *Then,*

$$\mathrm{hspec}^{\mathcal{S}}_{fg}(G) = [0, 1],$$

where $\mathcal{S}$ *is the congruence subgroups filtration.*

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Actually, this spectrum can be covered just using 3-geneator closed subgroups.

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Examples

- *(B. Klopsch, A. Thillaisundaram) The pro-p group* $W = C_p \,\hat{\wr}\, \mathbb{Z}_p$ *satisfies*

$$\begin{aligned} \mathrm{hspec}^{\mathcal{P}}_{fg}(W) = \mathrm{hspec}^{\mathcal{D}}_{fg}(W) &= \mathrm{hspec}^{\mathcal{F}}_{fg}(W) \\ &= \{m/p^n \mid n \in \mathbb{N}_0, 0 \leq m \leq p^n\}, \\ \mathrm{hspec}^{\mathcal{L}}_{fg}(W) = \{0\} &\cup \{1/2 + m/2p^n \mid n \in \mathbb{N}_0, 0 \leq m \leq p^n\}. \end{aligned}$$

Finitely generated Hausdorff spectrum

Problem

Let G be a finitely generated pro-p group and let $\mathcal{S} \in \{\mathcal{L}, \mathcal{D}, \mathcal{P}, \mathcal{I}, \mathcal{F}\}$. Is the finitely generated Hausdorff spectrum of G with respect to the filtration $\mathcal{S}$ discrete?

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Theorem (I.H., A. Thillaisundaram)

The groups $\mathfrak{G}(p)$ satisfy

$$\mathrm{hspec}_{fg}^{\mathcal{S}}(\mathfrak{G}(p)) = \{d^2/2^{2l} \mid 0 \leq d \leq 2^l, l \in \mathbb{N}_0\},$$

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So we do not know.

Eskerrik asko!

Grazie mille!