On unital near-trusses, dual weak braces and the set-theoretic Yang-Baxter equation

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The set-theoretic Yang-Baxter equation

Definition

A map $r : X \times X \to X \times X$ is a **solution of set-theoretic Yang-Baxter equation** if

$$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r).$$

- For $x, y \in X$, we write $r(x, y) = (\lambda_a(b), \rho_b(a))$.
- We say r is involutive if $r^2 = id$.
- We say that r is left (right) non-degenerate if for all $a \in X$ $\lambda_a(-) \quad (\rho_a(-))$ is a bijection

Skew braces

Definition

A **skew brace** is a triple $(B, +, \circ)$, where $(B, +)$ and $(B, \circ)$ are groups, and for all $a, b, c \in B$,

$$a \circ (b + c) = a \circ b - a + a \circ c$$

Definition

A **right distributor of a skew brace** B is a set

$$\mathcal{D}_r(B) = \{a \in B \mid (b + c) \circ a = b \circ a - a + c \circ a \; \forall b, c \in B\}.$$

- $\mathrm{Fix}(B) = \{a \in B \mid \lambda_x(a) = -x + x \circ a = a \forall x \in B\} \subseteq \mathcal{D}_r(B)$,
- $Z(B, \circ) \subseteq \mathcal{D}_r(B)$.

Examples of distributors

Example

Let $(B, +, \circ)$ be the brace on $(\mathbb{Z}, \circ) = \langle\, g \,\rangle$ with $g^k + g^l = g^{k+(-1)^k l}$, for all $k, l \in \mathbb{Z}$. Then, if $k, l, z \in \mathbb{Z}$, since

$$\left(g^k + g^l\right) \circ g^z = g^{k+z+(-1)^k l}$$

$$g^k \circ g^z - g^z + g^l \circ g^z = g^{k+z+(-1)^k (l+2z)}$$

it follows that $\mathcal{D}_r\,(B)$ is trivial.

Example

Let us consider the skew brace $(B, +, \circ)$ with trivial distributor and the brace $U_9 := \left(U\left(\mathbb{Z}/2^9\mathbb{Z}\right), +_1, \circ\right)$, with $a +_1 b = a - 1 + b$ and $a \circ b = ab$, for all $a, b \in U(\mathbb{Z}/2^9\mathbb{Z})$, with the usual operations in the ring modulo 2^9. Then, $U_9 \times B$ is a left skew brace such that $\mathcal{D}_r\,(U_9 \times B) = U_9 \times \{0\}$.

Solutions from skew braces

Theorem

Let B be a skew brace and $z \in \mathcal{D}_r(B)$, then the following maps

$$\check{r}_z(a,b) := (\check{\lambda}^z_a(b), \check{\tau}^z_b(a)) = (a\circ b - a\circ z + z, (a\circ b - a\circ z + z)^{-1}\circ a\circ b)$$

$$r_z(a,b) := (-a\circ z + a\circ b\circ z, (-a\circ z + a\circ b\circ z)^{-1}\circ a\circ b)$$

are non-degenerate solutions of the set-theoretic Yang-Baxter equation.

Remark

$$(\check{r}_z)^{-1} = r_{z^{-1}}$$

What next

We can go further and extend this connection for the following two cases:

- Unital near-trusses
- Dual weak braces

Near-trusses

Definition

A **unital near-truss** is a triple $(T, +, \cdot)$, such that $(T, +)$ is a group, $(T, \cdot, 1)$ is a monoid and for all $a, b, c \in T$,

$$a(b + c) = ab - a + ac.$$

Definition

A **homomorphism of unital near-trusses** is a map $f : T \to S$ such that $f(1) = 1$, $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$.

Remark

If $f : T \to S$ is a homomorphism of unital near-trusses, then

$$\mathrm{ker}_1(f) := f^{-1}(1) = \{a \in T \mid f(a) = 1\}$$

is a unital near-truss and $\mathrm{ker}_1(f) \subseteq T$.

Sequences

- B is a near-truss
- for any near-truss T, we can define $\check{\lambda}_a^z(b) := ab - az + z$, but it is not invertible.
- for any homomorphism of near-trusses $f : T \to S$,

$$f(\check{\lambda}_a^z(b)) = \check{\lambda}_{f(a)}^{f(z)}(f(b)).$$

If we consider the following sequence:

$$T \underset{\gamma}{\overset{\pi}{\rightleftarrows}} B \xrightarrow{t} \{1\}\ . \tag{1}$$

In this case, by exactness we understand $t^{-1}(1) = \mathrm{Im}(\pi)$ and $\pi\gamma = \mathrm{id}_B$, that is π is surjective and γ is injective.

Solution

Theorem

Let T be a unital near-truss and B a skew brace. Let $z \in T$ such that $\pi(z) \in \mathcal{D}_r(B)$ and suppose there exist two homomorphisms of unital near-trusses $\pi : T \to B$ and $\gamma : B \to T$ such that $\pi\gamma = \mathrm{id}_B$. Then, the map $\check{\eta} := \gamma\pi\check{\lambda}$ gives rise to the solution $\check{r}^z : T \times T \to T \times T$ defined by

$$\check{r}^z(a, b) := \left(\check{\eta}^z_a(b),\ (\check{\eta}^z_a(b))^{-1}\, ab\right),$$

for all $a, b \in T$.

Remarks

Remark

For every unital near-truss T, there exists the solution $\check{r}_{1_T}(a,b) = (1_T, ab)$, given by $B = \{1\}$, $a \overset{\pi}{\mapsto} 1$ and $\gamma(1) = 1_T$.

Remark

The restriction of $\check{r}_z : T \times T \to T \times T$ to $\check{r}_z|_{\gamma(B)\times\gamma(B)}$ gives an equivalent solution to the solution $\check{r}^{\pi(z)}$ on B via the bijection $\pi|_{\gamma(B)}$.

Remark

Let T be a unital near-truss and $a, z \in T$. The map $\check{\eta}_a^z : T \to T$ is injective if and only if $\pi : T \to B$ is a bijection.

Example

Let us consider the brace $(U(\mathbb{Z}/8\mathbb{Z}), +_1, \circ)$ on the units of the ring of integers modulo 8, $U(\mathbb{Z}/8\mathbb{Z})$, endowed with sandwich addition $a +_1 b := a - 1 + b$, for all $a, b \in U(\mathbb{Z}/8\mathbb{Z})$, and with the modulo multiplication. Let us consider the unital near-truss $\mathbb{Z}^8 := \mathrm{T}(U(\mathbb{Z}/8\mathbb{Z})) \times \mathbb{Z}$, $\pi : \mathbb{Z}^8 \to U(\mathbb{Z}/8\mathbb{Z}), (k, n) \mapsto k$, $\gamma : U(\mathbb{Z}/8\mathbb{Z}) \to \mathbb{Z}^8, k \mapsto (k, 1)$, and $z = (3, p)$ for some fixed $p \in \mathbb{Z}$. Then, the map $r^{(3,p)} : \mathbb{Z}^8 \times \mathbb{Z}^8 \to \mathbb{Z}^8 \times \mathbb{Z}^8$ defined by

$$r^{(3,p)}((k, n), (l, m)) = \Big((kl - 3k + 3, 1), ((kl - 3k + 3)^{-1} kl, nm)\Big)$$

is a deformed solution on $\mathbb{Z}^8$.

Dual weak braces

Definition

Let S be a set endowed with two binary operations $+$ and $\circ$ such that $(S,+)$ and $(S,\circ)$ are inverse semigroups. Then, $(S,+,\circ)$ is said to be a *weak (left) brace* if the following relations

$$a\circ(b+c)=a\circ b-a+a\circ c \qquad \& \qquad a\circ a^{-}=-a+a$$

are satisfied, for all $a,b,c\in S$, where $-a$ and a^{-} denote the inverses of a with respect to $+$ and $\circ$, respectively. Moreover, a weak brace $(S,+,\circ)$ is said to be a *dual weak brace* if $(S,\circ)$ is a Clifford semigroup.

In any weak brace the sets of idempotents $E(S,+)$ and $E(S,\circ)$ coincide, thus we simply denote them by $E(S)$. More in general, any idempotent $e\in E(S)$ satisfies the following

$$e+a=e\circ a=\lambda_e(a), \tag{2}$$

for every $a\in S$.

Examples

Example

Let $(S, \wedge)$ be a semilattice, $\{B_\alpha, +_\alpha, \circ_\alpha\}_{\alpha \in S}$ be a family of skew braces and let for all $\alpha \leqslant \beta$ exist $\mu_{\alpha,\beta} \in \mathrm{Hom}(B_\beta, B\alpha)$. Then we say that $[S, B_\alpha, \mu]$ is a strong semilattice of skew braces if

- for all $\alpha \in B_\alpha$ $\mu_{\alpha,\alpha} = id_\alpha$,
- for all $\alpha \leqslant \beta \leqslant \gamma$ $\mu_{\alpha,\beta}\mu_{\beta,\gamma} = \mu_{\alpha,\gamma}$.

If $[S, B_\alpha, \mu]$ is a strong semilattice, then the triple $B(S) := (\bigsqcup_{\alpha \in S} B_\alpha, +, \circ)$ is a dual weak brace, where for all $a \in B_\alpha$ and $b \in B_\beta$

$$a + b = \mu_{\alpha\wedge\beta,\alpha}(a) +_{\alpha\wedge\beta} \mu_{\alpha\wedge\beta,\beta}(b)$$
$$a \circ b = \mu_{\alpha\wedge\beta,\alpha}(a) \circ_{\alpha\wedge\beta} \mu_{\alpha\wedge\beta,\beta}(b)$$

Solutions again

Theorem

Let S be a dual weak brace and $z \in S$. Then, the map $r_z : S \times S \to S \times S$ given by

$$r_z(a, b) = \left(-a \circ z + a \circ b \circ z, (-a \circ z + a \circ b \circ z)^{-} \circ a \circ b\right),$$

for all $a, b \in S$, is a solution if and only if $z \in \mathcal{D}_r(S)$. We call such a map r_z solution associated to S deformed by z.

Solutions again

Proposition

Let $(S, +, \circ)$ be a dual weak brace and $z \in S$. Then, the function $\lambda^z : (S, \circ) \to \mathrm{Map}(S)$ is a homomorphism if and only if $a \circ z = z + a$, for every $a \in S$.

In the case of a dual weak brace S, even if $\lambda^z : (S, \circ) \to \mathrm{Map}(S)$ is a homomorphism, in general, λ_a^z does not coincide with λ_a, since $\lambda_a^z(b) = \lambda_a(b) + z \circ z^-$. In the study of deformed solutions, the following question arises.

Equivalence

Question 1
Let $(S, +, \circ)$ **be a dual weak brace. For which parameters** $z, w \in S$**, are the deformed solutions** r_z **and** r_w **equivalent?**

Equivalence

The following example shows that in the case of dual weak braces, even deformed solutions by idempotents are not equivalent in general.

Example

Let $X = \{e, x, y\}$ and $(S, \circ)$ the commutative inverse monoid on X with identity e satisfying the relations $x \circ x = y \circ y = x$ and $x \circ y = y$. Note that $a^- = a$, for every $a \in S$. Consider the trivial weak braces on S, namely $a + b = a \circ b$, for all $a, b \in S$. Then, we have two solutions $r_e = r$ and r_x related to the two idempotents e and x, respectively, for which the maps λ^e and λ^x are explicitly given by $\lambda^e_a(b) = \lambda_a(b) + e = \lambda_a(b) = a \circ a \circ b$ and $\lambda^x_a(b) = \lambda_a(b) + x = -a + a \circ b + x = a \circ a \circ b \circ x$. If the two solutions r_x and r_e were equivalent via a bijection $\varphi : S \to S$, then, in particular, we would have that $\varphi(a \circ a \circ b \circ x) = \varphi(a) \circ \varphi(a) \circ \varphi(b)$, for all $a, b \in S$. Thus, if $a = b = e$ we have that $\varphi(x) = \varphi(e) \circ \varphi(e) \circ \varphi(e) = \varphi(e)$, a contradiction.

Proposition

Let $(B, +, \circ)$ be a two-sided skew brace and $z, w \in B$ belonging to the same conjugacy class in $(B, \circ)$. Then, the deformed solutions r_z and r_w are equivalent.

Remark

Note that the converse of is not true. To show this, it is enough to consider the trivial brace $(B, +, +)$ on the cyclic group $\mathbb{Z}/2\mathbb{Z}$. Then, the solution r_0 coincides with the solution r_1 but 0 and 1 trivially belong to different conjugacy classes.

Summary:

- Right distributor is an interesting subset of a skew brace/near-truss.
- Solutions can be extended to near-trusses, although we loose invertibility and somehow trivialise part of a solution.
- The idea of a solution with a parameter can be extended to Dual weak braces.
- For which parameters solutions are equivalent?

Thank you