Detecting the nilpotency of a group from its non-commuting graph

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Advances in Group Theory and Applications 2023

June 6, 2023

Joint work with

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V. Grazian and C. M.,
A conjecture related to the nilpotency of groups with isomorphic non-commuting graphs, preprint available at arXiv:2302.01770 [math.GR] (2023)

The non-commuting graph

The non-commuting graph $\Gamma_{NC}(G) = (V, E)$ of a group G is defined as follows

- $V = G \setminus Z(G)$
- $\{x, y\} \in E \iff x$ and y do not commute.

This graph was firstly considered by Paul Erdös in 1975, when he posed a question then solved by Neumann.

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Combinatorial invariants of the graph

$$\Downarrow$$

Algebraic informations of the group

Graph isomorphism

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ denote two graphs.

We say that the two graphs Γ and Γ' are isomorphic if and only if

- there is a bijection ϕ between the sets of vertices V and V';
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What about the non-commuting graph

Conjecture A

Let G and H be finite groups such that $\Gamma_{NC}(G)$ and $\Gamma_{NC}(H)$ are isomorphic graphs. If G is nilpotent, then H is nilpotent as well.

This Conjecture was posed by **Abdollahi**, **Akbari**, and **Maimani** in

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Theorem (Abdollahi, Akbari, Maimani)

Let G and H be finite non-abelian groups with isomorphic non-commuting graphs.
If G is nilpotent and $|G| = |H|$, then H is nilpotent.

Proof.

★ By a result of **Cossey**, **Hawkes**, and **Mann**, it is enough to prove that G and H have the same number of conjugacy classes of the same size.

★ For every integer $i \geq 1$, let $m_i(G)$ and $m_i(H)$ denote the number of conjugacy classes of size i of G and H, respectively.

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★ For every integer $i \geq 1$, let $m_i(G)$ and $m_i(H)$ denote the number of conjugacy classes of size i of G and H, respectively.

★ Since $|G| - |Z(G)| = |H| - |Z(H)|$ and $|G| = |H|$, we have $m_1(G) = |Z(G)| = |Z(H)| = m_1(H)$.

★ Now, for every $g \in G \setminus Z(G)$, the number of vertices adjacent to g are $|G| - |C_G(g)|$.

★ Denote by h the image of g under the graph isomorphism.

★ Then $|G| - |C_G(g)| = |H| - |C_H(h)|$, which implies $|C_G(g)| = |C_H(h)|$.

★ As a consequence we obtain

$$|g^G| = |G : C_G(g)| = |H : C_H(h)| = |h^H|.$$

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A conjecture of **Abdollahi**, **Akbari**, and **Maimani**

Conjecture B (Abdollahi, Akbari, Maimani)

Let G and H be non-abelian finite groups with isomorphic non-commuting graphs. Then $|G| = |H|$.

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Clearly, if Conjecture B holds, then Conjecture A is true as well.

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The answer to Conjecture B is in the affirmative if

- G is a p-group.

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- G is a Dihedral group;

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A reduction theorem for Conjecture B

A group G is said to be an AC-group if every non central element has an abelian centralizer.

Theorem (Abdollahi, Akbari, Maimani)

If Conjecture B is true for AC-groups G and H, then it is true for all groups G and H.

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Theorem (Schmidt - 1970)

Let G be a **finite non-abelian solvable AC-group**. *Then either*

1. *G is non-nilpotent and it has an abelian normal subgroup N of prime index; or*
2. *$G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively, and F and K are abelian subgroups of G;or*
3. *$G/Z(G)$ is a Frobenius group with Frobenius kernel and complement $F/Z(G)$ and $K/Z(G)$, respectively, K is an abelian subgroup of G, $Z(F) = Z(G)$, and $F/Z(F)$ is of prime power order; or*
4. *$G/Z(G) \cong Sym(4)$ and V is a non-abelian subgroup of G such that $V/Z(G)$ is the Klein 4-group of $G/Z(G)$; or*
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A counter example to Conjecture B

In 2006, **Moghaddamfar** constructed two AC-groups:

★ $G = P \times A$, where P is a non-abelian 2-group of order 2^{10} and A an arbitrary abelian group;

★ $H = Q \times B$, where Q is a non-abelian 5-group of order 5^6 and B an arbitrary abelian group;

★ $|A| \cdot 2^5 = 4 \cdot |B| \cdot 5^3$.

Then $\Gamma_{NC}(G)$ and $\Gamma_{NC}(H)$ are isomorphic but $|G| \neq |H|$.

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Theorem (Abdollahi and Shahverdi)

Let **G** *be a finite non-abelian* **nilpotent** *group and suppose $\Gamma_{NC}(G)$ and $\Gamma_{NC}(H)$ are isomorphic, for a finite group H.*

If G has at least two distinct non-abelian Sylow subgroups and $|Z(G)| \geq |Z(H)|$ then $|G| = |H|$, and so H is nilpotent.

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Therefore, in order to prove Conjecture A, it is necessary to study the class of **finite non-abelian nilpotent groups having a unique non-abelian Sylow subgroup**, that is, finite groups G of the form $G = P \times A$, where p is a prime, P is the non-abelian Sylow p-subgroup of G and A is an abelian p'-group.

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A new conjecture

Together with Valentina Grazian, we conjecture the following:

Conjecture C

Let p be a prime and suppose $G = P \times A$ is a finite group where $P \in Syl_p(G)$ is non-abelian and A is an abelian p'-group.

If $\Gamma_{NC}(G)$ and $\Gamma_{NC}(H)$ are isomorphic for a finite group H and $|Z(G)| \geq |Z(H)|$ **then** $H = Q \times B$, where q is a prime, $Q \in Syl_q(H)$ is non-abelian and B is an abelian q'-group.

In particular, H **is nilpotent**.

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In particular, H **is nilpotent**.

Our contribution

Theorem (Grazian, M.)

Let $G = P \times A$ be a finite non-abelian **nilpotent AC-group** *and H be a group such that $\Gamma_{NC}(G)$ and $\Gamma_{NC}(H)$ are isomorphic.*

Then H **is a finite AC-group** *and either*

(a) $H = Q \times B$, *where q is a prime, $Q \in Syl_q(H)$ is non-abelian and B is an abelian q'-group; or*

(b) $|Z(H)| > |Z(G)|$, $|P : Z(P)| > p^4$, *none of the maximal subgroups of G is abelian and $H/Z(H)$ is a Frobenius group with Frobenius kernel and complement $F/Z(H)$ and $K/Z(H)$, respectively, where K is an abelian subgroup of H, $Z(F) = Z(H)$, $F/Z(H)$ is of prime power order q^t and the q-Sylow subgroups of G are abelian.*

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Let $G = P \times A$ be a finite non-abelian **nilpotent AC-group** *and H be a group such that $\Gamma_{NC}(G)$ and $\Gamma_{NC}(H)$ are isomorphic.*

Then H **is a finite AC-group** *and either*

(a) *$H = Q \times B$, where q is a prime, $Q \in Syl_q(H)$ is non-abelian and B is an abelian q'-group; or*

(b) *$|Z(H)| > |Z(G)|$, $|P : Z(P)| > p^4$, none of the maximal subgroups of G is abelian and $H/Z(H)$ is a Frobenius group with Frobenius kernel and complement $F/Z(H)$ and $K/Z(H)$, respectively, where K is an abelian subgroup of H, $Z(F) = Z(H)$, $F/Z(H)$ is of prime power order q^t and the q-Sylow subgroups of G are abelian.*

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Thank you for the attention

The problem we are interested in

Let $\Gamma(G)$ and $\Gamma(H)$ be graphs associated with the finite groups G and H.

Question A

Assume that $\Gamma(G)$ and $\Gamma(H)$ are isomorphic graphs, and G is nilpotent, is it true that H is nilpotent as well?

Joint work with

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V. Grazian, A. Lucchini and C. M.,
Group nilpotency from a graph point of view, preprint available at arXiv:2303.01093 [math.GR] (2023)

Graphs	Answers	If open: cases with positive answer; or counterexamples
Non-commuting graph	Open	YES if $\|G\| = \|H\|$ or G AC-group and $\|Z(G)\| \geq \|Z(H)\|$
Power graph	YES	
Prime graph	NO	$G \cong C_6 \times C_6$ and $H \cong S_3 \times C_6$ YES if $\|H\|$ is square-free
Generating graph	Open	YES if H supersoluble
Non-generating graph	Open	YES if remove all universal vertices and the subgraph is disconnected
Engel graph	YES	
Join graph	NO	$G \cong C_p \times C_p$, $H \cong D_{2p}$, $p > 2$ prime. H is proved to be supersoluble.

Table: Answers to Question A depending on the graphs.

Thank you again