What is the probability that two elements commute?

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Advances in Group Theory and Applications 2023

Joint work with A. Maróti and N. N. Hung

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- $|G|_\pi$ denotes the π-part of $|G|$.
- $k(G)$ denotes the number of conjugacy classes in G.
- $k_\pi(G)$ denotes the number of conjugacy classes of G whose elements are π-elements.

Let G be a finite group. We define the commuting probability of G as the probability that two random elements of G commute, that is

$$\Pr(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

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(I) Present results that give information on the structure of a group in terms of $\Pr(G)$.

(II) Generalise the indicator $\Pr(G)$ to obtain results on the existence of abelian and nilpotent Hall π-subgroups.

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With this result, Gustafson proved the following theorem.

Theorem (W.H. Gustafson, 1973)

Let G be a group. If $\Pr(G) > \frac{5}{8}$, *then G is abelian.*

Note

The bound above cannot be improved since Q_8 is a non-abelian group with $\Pr(Q_8) = \frac{5}{8}$.

The following result provides a bound for $\mathrm{Pr}(G)$ of non-abelian groups in terms of the smallest prime dividing $|G|$.

Theorem (T.Burness, R. M. Guralnick, A. Moretó and G. Navarro, 2021)

Let G be a group and let p be the smallest prime dividing $|G|$. If $\mathrm{Pr}(G) > \frac{p^2+p-1}{p^3}$, then G is abelian.

Note

Observe that $\frac{5}{8} = \frac{2^2+2-1}{2^3}$ and hence, this result generalises Gustafson's Theorem.

It is also possible to determine the nilpotency in terms of $\Pr(G)$.

Theorem (R. M. Guralnick and G. Robinson, 2005)

Let G be a group and let p be the smallest prime dividing $|G|$. If $\Pr(G) > \frac{1}{p}$, then G is nilpotent.

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This result was generalised in a recent work.

Theorem (N. N. Hung, A. Maróti and J.M., 2022)

Let G be a group and let p be the smallest prime dividing $|G|$. Then $\frac{1}{p} < \Pr(G) \leq \frac{p^2+p-1}{p^3}$ if and only if $|G'| = p$. Moreover, in such case,

$$\Pr(G) = \frac{1}{p} + \frac{p-1}{p|G : Z(G)|}.$$

Nilpotent and abelian Hall π-subgroups

Now, we turn our attention to results on the existence of nilpotent/abelian Hall π-subgroups. It is important to remark that these local results are much more complicated than the results on the structure of the whole group.

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Definition

Let G be a group and let π be a set of primes. We say that $G \in \mathcal{D}_\pi$ if G possesses Hall π-subgroups, all Hall π-subgroups are conjugated and every π-subgroup of G is contained in a Hall π-subgroup of G.

Theorem (H. Wielandt, 1954)

If G possesses a nilpotent Hall π-subgroup, then $G \in \mathcal{D}_\pi$.

First Generalization of $\Pr(G)$

Given a prime, p, Burness, Guralnick, Moretó and Navarro defined the following indicator.

Definition

Let G be a group and let p be a prime. We define the p-commuting probability of G as

$$\Pr_p(G) = \frac{|\{(x,y) \in G_p \times G_p | xy = yx\}|}{|G_p|^2}.$$

Theorem (T. Burness, R. Guralnick, A. Moretó and G. Navarro, 2021)

Let G be a group and let p be a prime. If $\Pr_p(G) > \frac{p^2+p-1}{p^3}$, then G possesses a normal and abelian Sylow p-subgroup.

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- The proof of the above result is very technical and it relies on deep results on actions of groups.
- The bound above cannot be improved for any prime $p > 2$. This is because the group $\mathrm{PSL}(2, p)$ is simple and $\mathrm{Pr}_p(\mathrm{PSL}(2, p)) = \frac{p^2+p-1}{p^3}$.

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- There is no analogue of the expression $\mathrm{Pr}(G) = \frac{k(G)}{|G|}$ for $\mathrm{Pr}_p(G)$.

Second Generalization of Pr(G)

In view of the equality $\Pr(G) = \frac{k(G)}{|G|}$, Hung and Maróti defined the following indicator.

Definition

Let G be a group and let π be a set of primes. We define

$$\mathrm{d}_\pi(G) = \frac{k_\pi(G)}{|G|_\pi}.$$

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Note

It is possible to prove that $\mathrm{d}_\pi(G) \leq 1$ for every group G and every set of primes π.

Using the indicator $\mathrm{d}_\pi(G)$, Hung and Maróti proved the following sufficient condition for the existence of abelian Hall π-subgroups

Theorem (N. N. Hung and A. Maróti, 2014)

Let G be a group and let π be a set of primes. If $\mathrm{d}_\pi(G) > \frac{5}{8}$, then G possesses abelian Hall π-subgroups.

Main result

Theorem (N. N. Hung, A. Maróti and J.M., 2022)

Let G be a group, let π be a set of primes and let p be the smallest prime in π.

(i) *If $\mathrm{d}_\pi(G) > \frac{p^2+p-1}{p^3}$, then G possesses abelian Hall π-subgroups.*

(ii) *If $\mathrm{d}_\pi(G) > \frac{1}{p}$, then G possesses nilpotent Hall π-subgroups.*

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We close this talk by showing how to reduce part (II) of our main result to simple groups.

Reduction to simple groups

To prove the reduction to simple groups, we need some preliminary results.

Lemma 1 (J. Fulman and R. Guralnick, 2012)

Let $N \trianglelefteq G$*. Then*

$$k_\pi(G) \leq k_\pi(N)k_\pi(G/N)$$

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Lemma 2 (P. Hall, 1956)

Let G *be a group and let* $N \trianglelefteq G$*. If both* G/N *and* N *possess nilpotent Hall* π*-subgroups, then* $G \in \mathcal{D}_\pi$*.*

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- *Since $|G|_\pi = |H|$, we have that*

$$\frac{1}{p} < \mathrm{d}_\pi(G) = \frac{k_\pi(G)}{|G|_\pi} \leq \frac{k(H)}{|H|} = \mathrm{Pr}(H).$$

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- *By Guralnick-Robinson's Theorem, we deduce that H is nilpotent.*

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- By Lemma 1, we have that $\mathrm{d}_\pi(G) \leq \mathrm{d}_\pi(N)\,\mathrm{d}_\pi(G/N)$ and hence $\frac{1}{p} < \mathrm{d}_\pi(G/N)$ and $\frac{1}{p} < \mathrm{d}_\pi(N)$.

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- By Lemma 2, $G \in \mathcal{D}_\pi$.
- By Lemma 3, G possesses nilpotent Hall π-subgroups, which is a contradiction.
- G must be simple as we wanted.

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$$\mathrm{d}_\pi(S) = \frac{k_\pi(S)}{|S|_\pi} \leq \frac{P_S(q)}{|S|_p}.$$

- It follows that $\mathrm{d}_\pi(S) < 1/p$ for all S unless for a finite list of examples and the result can be checked in these groups.

Thank you for your attention.

Any question?