Polynomial identities: some computational aspects

Daniela La Mattina

University of Palermo

Advances in Group Theory and Applications 2023, June 6th-9th 2023

Definition

A polynomial identity of an algebra A is a polynomial in non-commuting indeterminates vanishing under all evaluations in A.

Definition

A polynomial identity of an algebra A is a polynomial in non-commuting indeterminates vanishing under all evaluations in A.

Definition

A polynomial identity of an algebra A is a polynomial in non-commuting indeterminates vanishing under all evaluations in A.

$xy - yx$ is a polynomial identity for any commutative algebra.

A bit of history

- Dehn, M.: Über die Grundlagen der projektiven Geometrie und allgemeine Zahlsysteme. Math. Ann. 85 (1922);

A bit of history

- Dehn, M.: Über die Grundlagen der projektiven Geometrie und allgemeine Zahlsysteme. Math. Ann. 85 (1922);

- Wagner W.: Über die Grundlagen der projektiven Geometrie und allgemeine Zahlensysteme Math. Ann. (1937);

A bit of history

- Dehn, M.: Über die Grundlagen der projektiven Geometrie und allgemeine Zahlsysteme. Math. Ann. 85 (1922);

- Wagner W.: Über die Grundlagen der projektiven Geometrie und allgemeine Zahlensysteme Math. Ann. (1937);

- Kaplansky I.: Rings with a polynomial identity. Bull. Amer. Math. Soc. 54 (1948).

A bit of history

Positive answer to the Kurosh problem for PI-algebras

A bit of history

Positive answer to the Kurosh problem for PI-algebras

Kurosh problem, 1941: Is every finitely generated algebraic algebra finite dimensional?

Counterexamples: Golod-Shafarevich, 1964.

Combinatorial approach in PI-theory

Theorem (Amitsur, Levitzki, 1950)

The standard polynomial of degree $2n$

$$St_{2n}(x_1,\ldots,x_{2n}) = \sum_{\sigma\in S_{2n}} (sgn\sigma) x_{\sigma(1)}\cdots x_{\sigma(2n)}$$

is a polynomial identity of minimal degree for $M_n(F)$ *the algebra of* $n\times n$ *matrices.*

$F\langle X \rangle =$ the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over a field F of characteristic zero

$F\langle X \rangle$ = the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over a field F of characteristic zero

A = associative F-algebra

$F\langle X\rangle =$ the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over a field F of characteristic zero

$A =$ associative F-algebra

$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in } A\}$ = the T-ideal of $F\langle X\rangle$ of polynomial identities of A

$F\langle X\rangle =$ the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over a field F of characteristic zero

$A =$ associative F-algebra

$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in } A\}$ = the T-ideal of $F\langle X\rangle$ of polynomial identities of A

Specht Problem (1950): Is every proper T-ideal of the free associative algebra finitely generated?

$F\langle X\rangle$ = the free associative algebra on a countable set $X = \{x_1, x_2, \dots\}$ over a field F of characteristic zero

A = associative F-algebra

$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in } A\}$ = the T-ideal of $F\langle X\rangle$ of polynomial identities of A

Specht Problem (1950): Is every proper T-ideal of the free associative algebra finitely generated?

$F\langle X\rangle =$ the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ over a field F of characteristic zero

$A =$ associative F-algebra

$\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in } A\}$ = the T-ideal of $F\langle X\rangle$ of polynomial identities of A

Specht Problem (1950): Is every proper T-ideal of the free associative algebra finitely generated?

Kemer (1987): Positive answer to the Specht problem.

Examples of T-ideals

- $\mathrm{Id}(F) = \langle [x_1, x_2] \rangle_T$, where $[x_1, x_2] = x_1x_2 - x_2x_1$;

Examples of T-ideals

- $\mathrm{Id}(F) = \langle [x_1, x_2] \rangle_T$, where $[x_1, x_2] = x_1x_2 - x_2x_1$;

- $M_2(F) =$ the algebra of 2×2 matrices over F,

$$\mathrm{Id}(M_2(F)) = \langle [[x_1, x_2]^2, x_3], St_4(x_1, x_2, x_3, x_4) \rangle_T.$$

If $\operatorname{char} F = 0$, $\operatorname{Id}(A)$ is completely determined by its multilinear polynomials.

If $\operatorname{char} F = 0$, $\operatorname{Id}(A)$ is completely determined by its multilinear polynomials.

For every $n \geq 1$, let

$$P_n = \operatorname{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

be the vector space of multilinear polynomials in $x_1, \ldots, x_n$.

If $\operatorname{char} F = 0$, $\operatorname{Id}(A)$ is completely determined by its multilinear polynomials.

For every $n \geq 1$, let

$$P_n = \operatorname{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

be the vector space of multilinear polynomials in $x_1, \ldots, x_n$.

$\operatorname{Id}(A)$ is generated by $P_n \cap \operatorname{Id}(A)$, $n \geq 1$.

Definition

The non-negative integer

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)},\ n \geq 1$$

is called the n-th codimension of A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the *codimension sequence* of A.

Definition

The non-negative integer

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)},\ n \geq 1$$

is called the n-th codimension of A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the *codimension sequence* of A.

Definition

The non-negative integer

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}, \; n \geq 1$$

is called the n-th codimension of A. The sequence $\{c_n(A)\}_{n\geq 1}$ is the *codimension sequence* of A.

Regev (1972) If A is a PI-algebra , then there exists $d \geq 1$ such that $c_n(A) \leq d^n$, for all n.

Codimension sequence of PI-algebras

- $c_n(F) = 1$, for all $n \geq 1$;

Codimension sequence of PI-algebras

- $c_n(F) = 1$, for all $n \geq 1$;

- $c_n(M_k(F)) \simeq Cn^{-\frac{1}{2}(k^2-1)}k^{2n}$, with C a constant (Regev, 1984).

In the 1980's two main conjectures were made:

In the 1980's two main conjectures were made:

Amitsur's conjecture: For any PI-algebra A, $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer.

In the 1980's two main conjectures were made:

Amitsur's conjecture: For any PI-algebra A, $\lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer.

Regev's conjecture: For any PI-algebra A, there exist a constant C, a semi-integer q and an integer $d \geq 0$ such that

$$c_n(A) \simeq C n^q d^n.$$

Amitsur's conjecture

Giambruno-Zaicev (1999):

For a PI-algebra A, there exist constants $c_1 > 0, c_2, k_1, k_2$, and an integer $d \geq 0$ such that

$$c_1 n^{k_1} d^n \leq c_n(A) \leq c_2 n^{k_2} d^n.$$

Hence $exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer called the exponent of A.

Regev's Conjecture

Berele-Regev (2008): For any PI-algebra A with 1, $c_n(A) \simeq Cn^t exp(A)^n$, with $t \in \frac{1}{2}\mathbb{Z}$.

Regev's Conjecture

Berele-Regev (2008): For any PI-algebra A with 1, $c_n(A) \simeq Cn^t exp(A)^n$, with $t \in \frac{1}{2}\mathbb{Z}$.

Giambruno-Zaicev(2014): $A = PI$-algebra. Then the sequence $c_n(A)$ is eventually non decreasing, i.e., $c_{n+1}(A) \geq c_n(A)$, for n large enough.

Regev's Conjecture

Berele-Regev (2008): For any PI-algebra A with 1, $c_n(A) \simeq Cn^t exp(A)^n$, with $t \in \frac{1}{2}\mathbb{Z}$.

Giambruno-Zaicev(2014): $A = PI$-algebra. Then the sequence $c_n(A)$ is eventually non decreasing, i.e., $c_{n+1}(A) \geq c_n(A)$, for n large enough.

For any PI-algebra A

$$C_1 n^t exp(A)^n \leq c_n(A) \leq C_2 n^t exp(A)^n, \quad t \in \frac{1}{2}\mathbb{Z}.$$

Regev's Conjecture

Berele-Regev (2008): For any PI-algebra A with 1, $c_n(A) \simeq Cn^t exp(A)^n$, with $t \in \frac{1}{2}\mathbb{Z}$.

Giambruno-Zaicev(2014): $A = PI$-algebra. Then the sequence $c_n(A)$ is eventually non decreasing, i.e., $c_{n+1}(A) \geq c_n(A)$, for n large enough.

For any PI-algebra A

$$C_1 n^t exp(A)^n \leq c_n(A) \leq C_2 n^t exp(A)^n, \quad t \in \frac{1}{2}\mathbb{Z}.$$

Definition

$$t = pol(A) = \lim_{n\to\infty} \log_n \frac{c_n(A)}{exp(A)^n}.$$

Algebras with G-action: superalgebras and algebras with involution

A is an algebra with G-action where $G = \langle \varphi \rangle, \varphi$ is an automorphism or antiautomorphism of A of order ≤ 2.

Algebras with G-action: superalgebras and algebras with involution

A is an algebra with G-action where $G = \langle \varphi \rangle, \varphi$ is an automorphism or antiautomorphism of A of order ≤ 2.

$A = A_0^{\varphi} \oplus A_1^{\varphi}$, where

$$A_0^{\varphi} = \{a \in A \mid a^{\varphi} = a\} \text{ and } A_1^{\varphi} = \{a \in A \mid a^{\varphi} = -a\}.$$

Algebras with G-action: superalgebras and algebras with involution

A is an algebra with G-action where $G = \langle \varphi \rangle, \varphi$ is an automorphism or antiautomorphism of A of order ≤ 2.

$A = A_0^{\varphi} \oplus A_1^{\varphi}$, where

$$A_0^{\varphi} = \{a \in A \mid a^{\varphi} = a\} \text{ and } A_1^{\varphi} = \{a \in A \mid a^{\varphi} = -a\}.$$

if φ is an antiautomorphism (involution)

$A_0^{\varphi} = A^{+}$ and $A_1^{\varphi} = A^{-}$ are the subspaces of symmetric and skew elements, respectively.

Algebras with G-action: superalgebras and algebras with involution

A is an algebra with G-action where $G = \langle \varphi \rangle, \varphi$ is an automorphism or antiautomorphism of A of order ≤ 2.

$A = A_0^{\varphi} \oplus A_1^{\varphi}$, where

$$A_0^{\varphi} = \{a \in A \mid a^{\varphi} = a\} \text{ and } A_1^{\varphi} = \{a \in A \mid a^{\varphi} = -a\}.$$

if φ is an antiautomorphism (involution)

$A_0^{\varphi} = A^+$ and $A_1^{\varphi} = A^-$ are the subspaces of symmetric and skew elements, respectively.

if φ is an automorphism $\Rightarrow A$ is a $\mathbb{Z}_2$-graded algebra (superalgebra) with grading $(A^{(0)}, A^{(1)})$, where $A^{(0)} = A_0^{\varphi}$ and $A^{(1)} = A_1^{\varphi}$.

Algebras with G-action: superalgebras and algebras with involution

A is an algebra with G-action where $G = \langle \varphi \rangle, \varphi$ is an automorphism or antiautomorphism of A of order ≤ 2.

$A = A_0^{\varphi} \oplus A_1^{\varphi}$, where

$$A_0^{\varphi} = \{a \in A \mid a^{\varphi} = a\} \text{ and } A_1^{\varphi} = \{a \in A \mid a^{\varphi} = -a\}.$$

if φ is an antiautomorphism (involution)

$A_0^{\varphi} = A^+$ and $A_1^{\varphi} = A^-$ are the subspaces of symmetric and skew elements, respectively.

if φ is an automorphism $\Rightarrow A$ is a $\mathbb{Z}_2$-graded algebra (superalgebra) with grading $(A^{(0)}, A^{(1)})$, where $A^{(0)} = A_0^{\varphi}$ and $A^{(1)} = A_1^{\varphi}$.

A is called a φ-algebra

$F\langle X,\varphi\rangle = F\langle x_1, x_1^{\varphi}, x_2, x_2^{\varphi}, \ldots\rangle$ = the free associative algebra on X with G-action.

$F\langle X, \varphi \rangle = F\langle x_1, x_1^{\varphi}, x_2, x_2^{\varphi}, \ldots \rangle$ = the free associative algebra on X with G-action.

$\mathrm{Id}^{\varphi}(A)$= the set of φ-identities of A= T^{φ}-ideal of $F\langle X, \varphi \rangle$.

$F\langle X,\varphi\rangle = F\langle x_1, x_1^{\varphi}, x_2, x_2^{\varphi}, \ldots\rangle$ = the free associative algebra on X with G-action.

$\mathrm{Id}^{\varphi}(A)$= the set of φ-identities of A= T^{φ}-ideal of $F\langle X,\varphi\rangle$.

$F\langle X,\varphi\rangle = F\langle x_1, x_1^\varphi, x_2, x_2^\varphi, \dots\rangle$ = the free associative algebra on X with G-action.

$\mathrm{Id}^\varphi(A)$= the set of φ-identities of A= T^φ-ideal of $F\langle X,\varphi\rangle$.

For every $n \geq 1$, the non negative integer

$$c_n^\varphi(A) = \dim_F \frac{P_n^\varphi}{P_n^\varphi \cap Id^\varphi(A)}$$

is called the n-th φ-codimension of A, where P_n^φ is the vector space of multilinear polynomials of degree n in $x_1, x_1^\varphi, \dots, x_n, x_n^\varphi$.

$F\langle X,\varphi\rangle = F\langle x_1, x_1^{\varphi}, x_2, x_2^{\varphi}, \dots\rangle$ = the free associative algebra on X with G-action.

$\mathrm{Id}^{\varphi}(A)$= the set of φ-identities of A= T^{φ}-ideal of $F\langle X,\varphi\rangle$.

For every $n \geq 1$, the non negative integer

$$c_n^{\varphi}(A) = \dim_F \frac{P_n^{\varphi}}{P_n^{\varphi} \cap Id^{\varphi}(A)}$$

is called the n-th φ-codimension of A, where P_n^{φ} is the vector space of multilinear polynomials of degree n in $x_1, x_1^{\varphi}, \dots, x_n, x_n^{\varphi}$.

Giambruno-Regev (1985) If A is a PI-algebra then $c_n^{\varphi}(A)$, $n = 1, 2, \dots,$ is exponentially bounded.

Amitsur's conjecture for φ-algebras

Amitsur's conjecture: For any PI-φ-algebra A, $\lim_{n\to\infty} \sqrt[n]{c_n^{\varphi}(A)}$ exists and is a non-negative integer.

Amitsur's conjecture for φ-algebras

Amitsur's conjecture: For any PI-φ-algebra A, $\lim_{n\to\infty} \sqrt[n]{c_n^{\varphi}(A)}$ exists and is a non-negative integer.

Aljadeff-Giambruno-LM (2010-2011), Giambruno-Polcino-Valenti (2017):

For a PI-algebra A, there exist constants $c_1 > 0, c_2, k_1, k_2$, and an integer $d \geq 0$ such that

$$c_1 n^{k_1} d^n \leq c_n^{\varphi}(A) \leq c_2 n^{k_2} d^n.$$

Hence $exp^{\varphi}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\varphi}(A)}$ exists and is a non-negative integer called the φ-exponent of A.

Regev's conjecture for φ-algebras

Regev's conjecture: There exist a constant C, a semi-integer q and an integer $d \geq 0$ such that

$$c_n^{\varphi}(A) \simeq Cn^q d^n.$$

Finite dimensional simple algebras with involution

Berele, Giambruno, Regev (1996): If $A = (M_k(F), \varphi)$, $\varphi = t$ or s,

$$c_n^{\varphi}(A) \simeq C n^{-\frac{1}{2}\dim A_1^{\varphi}} (\dim A)^n$$

Finite dimensional simple algebras with involution

Berele, Giambruno, Regev (1996): If $A = (M_k(F), \varphi)$, $\varphi = t$ or s,

$$c_n^{\varphi}(A) \simeq Cn^{-\frac{1}{2}\dim A_1^{\varphi}} (\dim A)^n$$

Giambruno, Polcino Milies, L.M. (2020): If $A = M_k(F) \oplus M_k(F)^{op}$,

$$c_n^{\varphi}(A) \simeq Cn^{-\frac{1}{2}(\dim A_1^{\varphi} - 1)} (\dim A)^n$$

where $A_1^{\varphi} = A^{-} = \{a \in A \mid a^{\varphi} = -a\}$.

Finite dimensional simple superalgebras

Karasik-Shpigelman (2016) : If A is a finite dimensional φ-simple algebra then

$$c_n^{\varphi}(A) \simeq C n^{-\frac{1}{2}(\dim A_0^{\varphi} - 1)} (\dim A)^n,$$

for some constant C, where A_0^{φ} is the homogeneous component of degree zero.

For a PI-algebra A, there exist constants $c_1 > 0, c_2, k_1, k_2$, and an integer $d \geq 0$ such that

$$c_1 n^{k_1} d^n \leq c_n^{\varphi}(A) \leq c_2 n^{k_2} d^n.$$

For a PI-algebra A, there exist constants $c_1 > 0, c_2, k_1, k_2$, and an integer $d \geq 0$ such that

$$c_1 n^{k_1} d^n \leq c_n^{\varphi}(A) \leq c_2 n^{k_2} d^n.$$

Is $k_1 = k_2$?

For a PI-algebra A, there exist constants $c_1 > 0, c_2, k_1, k_2$, and an integer $d \geq 0$ such that

$$c_1 n^{k_1} d^n \leq c_n^{\varphi}(A) \leq c_2 n^{k_2} d^n.$$

Is $k_1 = k_2$?

Positive answer for finitely generated φ-algebras: $k_1 = k_2 \in \frac{1}{2}\mathbb{Z}$.

φ-fundamental algebras

Let A be a finite dimensional φ-algebra over $F = \bar{F}$.

φ-fundamental algebras

Let A be a finite dimensional φ-algebra over $F = \bar{F}$.

Let $A = \bar{A} + J$ where $\bar{A} = A_1 \oplus \cdots \oplus A_q$, with $A_1, \ldots, A_q$ simple φ-algebras, $J = J(A)$ the Jacobson radical of A

φ-fundamental algebras

Let A be a finite dimensional φ-algebra over $F = \bar{F}$.

Let $A = \bar{A} + J$ where $\bar{A} = A_1 \oplus \cdots \oplus A_q$, with $A_1, \ldots, A_q$ simple φ-algebras, $J = J(A)$ the Jacobson radical of A

$t = \dim \bar{A}$, $s \geq 0$ is the least integer such that $J^{s+1} = 0$.

φ-fundamental algebras

Let A be a finite dimensional φ-algebra over $F = \bar{F}$.

Let $A = \bar{A} + J$ where $\bar{A} = A_1 \oplus \cdots \oplus A_q$, with $A_1, \ldots, A_q$ simple φ-algebras, $J = J(A)$ the Jacobson radical of A

$t = \dim \bar{A}$, $s \geq 0$ is the least integer such that $J^{s+1} = 0$.

Definition

A is φ-fundamental if $\forall \mu \geq 1$ $\exists$ a multilinear φ-polynomial $f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_s, Y) \notin Id^\varphi(A)$ alternating in the μ sets X_i with $|X_i| = t$ and in the s sets Z_j with $|Z_j| = t + 1$.

φ-fundamental algebras

Theorem (Giambruno, Polcino Milies, LM (2020))

Let $A = \bar{A} + J$ be a φ-fundamental algebra, where φ is an antiautomorphism. Write $\bar{A} = A_1 \oplus \cdots A_r \oplus A_{r+1} \oplus \cdots \oplus A_q$, a direct sum of φ-simple algebras with $A_1, \ldots, A_r$ not simple algebras, then

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s} d^n \leq c_n^{\varphi}(A) \leq C_2 n^{-\frac{1}{2}(\dim(\bar{A})^- - r)+s} d^n,$$

for some constants $C_1 > 0, C_2$, where $s \geq 0$ is the least integer such that $J^{s+1} = 0$ and $d = exp^{\varphi}(A) = \dim \bar{A}$. Hence

$$\lim_{n\to\infty} \log_n \frac{c_n^{\varphi}(A)}{exp^{\varphi}(A)^n} = -\frac{1}{2}(\dim(\bar{A})^- - r) + s.$$

φ-fundamental algebras

Theorem (Giambruno, LM (2022))

Let $A = \bar{A} + J$ *be a* φ*-fundamental algebra, where* φ *is an automorphism. Write* $\bar{A} = A_1 \oplus \cdots \oplus A_q$*, a direct sum of* φ*-simple algebras then*

$$C_1 n^{-\frac{1}{2}(\dim(\bar{A})^{(0)} - q) + s} d^n \leq c_n^{\varphi}(A) \leq C_2 n^{-\frac{1}{2}(\dim(\bar{A})^{(0)} - q) + s} d^n,$$

for some constants $C_1 > 0, C_2$*, where* $s \geq 0$ *is the least integer such that* $J^{s+1} = 0$ *and* $d = exp^{\varphi}(A) = \dim \bar{A}$*. Hence*

$$\lim_{n \to \infty} \log_n \frac{c_n^{\varphi}(A)}{exp^{\varphi}(A)^n} = -\frac{1}{2}(\dim(\bar{A})^{(0)} - q) + s.$$

Finitely generated φ-algebras

remark

Every finite dimensional φ-algebra has the same φ-identities as a finite direct sum of φ-fundamental algebras.

Finitely generated φ-algebras

remark

Every finite dimensional φ-algebra has the same φ-identities as a finite direct sum of φ-fundamental algebras.

Finitely generated φ-algebras

remark

Every finite dimensional φ-algebra has the same φ-identities as a finite direct sum of φ-fundamental algebras.

A finitely generated PI φ-algebra has the same identities as a finite dimensional φ-algebra.

Theorem (Giambruno, Polcino Milies, LM, 2020, 2022)

Let A be a finitely generated PI φ-algebra over a field F of characteristic zero. Then

$$C_1 n^t exp^{\varphi}(A)^n \leq c_n^{\varphi}(A) \leq C_2 n^t exp^{\varphi}(A)^n,$$

where $t \in \frac{1}{2}\mathbb{Z}$, for some constants $C_1 > 0, C_2$.

Hence $\lim_{n\to\infty} \log_n \dfrac{c_n^{\varphi}(A)}{exp^(A)^n}$ exists and is a half integer.

Thank you for your attention!