Hopf-Galois Structures and Hunting Through the Holomorph

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Review of Galois Theory

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$$\mathrm{Gal}(L/\mathbb{Q}) := \{\sigma \in \mathrm{Aut}(L) \mid \sigma(x) = x \ \forall x \in \mathbb{Q}\}$$

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Theorem (Fundamental Theorem of Galois Theory)
If $L/\mathbb{Q}$ is Galois, then there is a bijective correspondence between

Fields $\mathbb{Q} < F < L$, and

Subgroups $H < \mathrm{Gal}(L/\mathbb{Q})$

given by $F = L^H$.

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This gives an example of a **Hopf-Galois Structure**

Some facts

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My work focuses on studying, describing and counting Hopf-Galois structures for different field extensions.

Back to group theory

Define the *holomorph*, $\mathrm{Hol}(N)$ of a group N to be the semidirect product of N and $\mathrm{Aut}(N)$:

$$Hol(N) \cong N \rtimes \mathrm{Aut}(N).$$

Where

$$(\eta, \alpha)(\mu, \beta) = (\eta\alpha(\mu), \alpha\beta).$$

Note: $\mathrm{Hol}(N)$ has a natural action on N given by:

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$$H = E[N]^G$$

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- $L/\mathbb{Q}$ degree p^2, $2p$ [CS20], mp with $(m, p) = 1$ [Koh07] & [Koh16], squarefree Galois [AB20],...

$L/\mathbb{Q}$ degree pq

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$$\mathrm{Hol}(C_{pq}) \cong C_{pq} \rtimes (C_{p-1} \times C_{q-1})$$
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In each case, we find the smallest subgroups of $\mathrm{Hol}(N)$ which are transitive on N and then build up.

For $N \cong C_{pq}$, these 'minimally transitive' subgroups are

$$N, \\ \langle \sigma, [\tau, \alpha^u] \rangle$$

for α generating the unique Sylow q-subgroup of $\mathrm{Aut}(N)$ and $u \neq 0$.

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For $N \cong C_p \rtimes C_q$, it is possible to write $\mathrm{Hol}(N)$ as $P \rtimes R$ for P, R abelian groups of orders $p^2, q(p-1)$ respectively.

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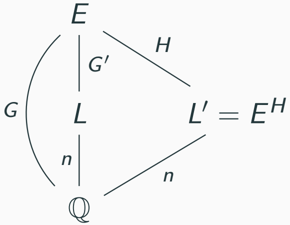
- 

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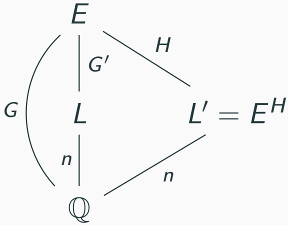
- 

  What can we say about $L'/\mathbb{Q}$?
- How much can we push these results to other related constructions?

Thank You!

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