LEFT SEMI-BRACES AND SOLUTIONS TO THE YANG-BAXTER EQUATION

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YANG-BAXTER AND ALGEBRAIC STRUCTURES

Definition

A set-theoretic solution to the Yang-Baxter equation is a tuple (X, r), where X is a set and $r : X \times X \longrightarrow X \times X$ a function such that (on X^3)

$$(\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r) = (r \times \mathrm{id}_X)\,(\mathrm{id}_X \times r)\,(r \times \mathrm{id}_X)\,.$$

For further reference, denote $r(x, y) = (\lambda_x(y), \rho_y(x))$.

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Definition

A set-theoretic solution (X, r) is called

- left (resp. right) non-degenerate, if λ_x (resp. ρ_y) is bijective,
- non-degenerate, if it is both left and right non-degenerate,
- involutive, if $r^2 = \mathrm{id}_{X \times X}$.

BRACES AND GENERALIZATIONS

Definition (Rump(1), CJO, GV (2))

A triple $(A, \cdot, \circ)$ is called a skew left brace, if $(A, \cdot)$ is a group and $(A, \circ)$ is a group such that for any $a, b, c \in A$,

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c),$$

where a^{-1} denotes the inverse of a in $(A, \cdot)$. In particular, if $(A, \cdot)$ is an abelian group, then $(A, \cdot, \circ)$ is called a left brace.

BRACES AND GENERALIZATIONS

Definition

A group $(A, \cdot)$ with additional group structure $(A, \circ)$ such that

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{-1} \cdot (a \circ c).$$

Definition (Catino, Colazzo, Stefanelli (3))

A triple $(B, \cdot, \circ)$ is called a left cancellative left semi-brace, if $(B, \cdot)$ is a left cancellative semi-group and $(B, \circ)$ is a group such that for any $a, b, c \in B$,

$$a \circ (b \cdot c) = (a \circ b) \cdot (a \circ (\overline{a} \cdot c)),$$

where $\overline{a}$ denotes the inverse of a in $(B, \circ)$.

STRUCTURE MONOID AND GROUP

Definition

Let (X, r) be a set-theoretic solution of the Yang-Baxter equation. Then the monoid

$$M(X, r) = \langle x \in X \mid xy = \lambda_x(y)\rho_y(x) \rangle ,$$

is called the structure monoid of (X, r).

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is called the structure monoid of (X, r). The group $G(X, r)$ generated by the same presentation is called the structure group of (X, r).

FROM YB TO BRACES

Theorem (ESS, LYZ, S, GV)

Let (X, r) be a non-degenerate solution to YBE, then there exists a unique skew left brace structure on $G(X, r)$ such that the associated solution r_G satisfies

$$r_G(i \times i) = (i \times i)r,$$

where $i : X \to G(X, r)$ is the canonical map.

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where $i : X \to G(X, r)$ is the canonical map. Moreover, if (X, r) is involutive, then $G(X, r)$ is a left brace and

$$r_G|_{X \times X} = r.$$

FROM BRACES TO YB

Definition

Let $(B, \cdot, \circ)$ be a skew left brace. Define $\lambda_a(b) = a^{-1}(a \circ b)$ and $\rho_b(a) = \overline{(\overline{a} \cdot b)} \circ b$. Then, $r_B(a,b) = (\lambda_a(b), \rho_a(b))$ is a bijective non-degenerate solution to YB.

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LEFT SEMI-BRACES

Definition

Let $(B, \cdot, \circ)$ be a triple such that $(B, \cdot)$ is a semi-group and $(B, \circ)$ is a group. If, for any $a, b, c \in B$, it holds that

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then this triple is called a left semi-brace.

Moreover, if $(B, \cdot)$ is left cancellative, then $(B, \cdot, \circ)$ is called a left cancellative left semi-brace. This is a left semi-brace in the sense of Catino, Colazzo and Stefanelli.

COMPLETELY SIMPLE

Definition

Let G be a group, I, J sets and $P = (p_{ji})$ a $|J| \times |I|$-matrix with entries in G. Then

$$\mathcal{M}(G, I, J, P) = \{(g, i, j) \mid g \in G, i \in I, j \in J\},$$

is called the Rees matrix semi-group associated to (G, I, J, P), where multiplication is defined as $(g, i, j)(h, k, l) = (g p_{jk} h, i, l)$.

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Theorem

Let S be a finite semi-group such that S has no non-trivial ideals and every idempotent of S is primitive (i.e. S is completely simple), then S is isomorphic to a Rees matrix semi-group. Conversely, every finite Rees matrix semi-group satisfies these conditions.

FINITE SEMI-BRACES

Theorem

Let $(B, \cdot, \circ)$ be a finite left semi-brace. Then $(B, \cdot)$ is completely simple. Moreover, there exists a finite group G and finite sets I, J such that $(B, \cdot) \cong \mathcal{M}(G, I, J, \mathcal{I}_{J,I})$, where $\mathcal{I}_{J,I}$ is the $J \times I$-matrix where every entry is 1. Furthermore, $(G, \cdot, \circ)$ is a skew left brace.

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Proposition

Let $(B, \cdot, \circ)$ be a left semi-brace. Then, the map $\lambda_a : B \to B : b \mapsto a \circ (\overline{a}b)$ is an endomorphism of $(B, \cdot)$. Furthermore, $\lambda : (B, \circ) \to \mathrm{End}(B, \cdot)$ is a semi-group morphism.

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Define for any $a, b \in B$, the map $\rho_b(a) = \overline{(\overline{a}b)} \circ b$.

THE ρ-CONDITION AND SOLUTIONS

Proposition

Let $(B, \cdot, \circ)$ be a left semi-brace. If $\rho : (B, \circ) \to \mathrm{Map}(B, B)$ is a semi-group anti-morphism, then $r_B(a, b) = (\lambda_a(b), \rho_b(a))$ is a set-theoretic solution to YB.

Not every left semi-brace satisfies this condition. However, is ρ-condition necessary?

THE CONDITION IN EQUATIONS

Proposition

Let $(B, \cdot, \circ)$ *be a left semi-brace. TFAE*

(1) $\rho : (B, \circ) \longrightarrow \mathrm{Map}(B, B)$ *is an anti-homomorphism.*

(2) $c\,(a \circ (1_\circ b)) = c\,(a \circ b)$ *for all* $a, b, c \in B$.

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(3) $(B, \cdot)$ *is completely simple and, for any* $(g, i, j) \in B$ *and* $(1, k, l) \in E(B)$, *if* $(h, r, s) = (g, i, j) \circ (1, k, l)$, *then* $h = g$.

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(3) $(B, \cdot)$ *is completely simple and, for any* $(g, i, j) \in B$ *and* $(1, k, l) \in E(B)$*, if* $(h, r, s) = (g, i, j) \circ (1, k, l)$*, then* $h = g$.

Moreover, in these cases, the idempotents $E(B)$ *form a left subsemi-brace as well as the idempotents* $E(B1_\circ)$ *of the left subsemi-brace* $B1_\circ$.

THE CONDITION IN STRUCTURE

Theorem

Let $(B, \cdot, \circ)$ be a left semi-brace. The following conditions are equivalent.

1. *ρ is an anti-homomorphism,*
2. *$B \cong (1_\circ B 1_\circ \bowtie E(B1_\circ)) \bowtie E(1_\circ B)$ and $E(B)$ is a left subsemi-brace of B.*

ALGEBRA OF STRUCTURE MONOID

Proposition

Let $(B, \cdot, \circ)$ be a left semi-brace such that ρ is an anti-homomorphism. Then, for any field K, the algebra $KM(B)$ is generated as a left (and right) $KM(1_\circ B 1_\circ)$-module by $(1_\circ B) * (B 1_\circ)$.

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Theorem

Let $(B, \cdot, \circ)$ be a finite left semi-brace such that ρ is an anti-homomorphism. Then, $KM(B)$ is a Noetherian, PI-algebra of finite Gelfand-Kirillov dimension equal to that of $KM(1_\circ B 1_\circ)$. In particular, this dimension is at most $|1_\circ B 1_\circ|$ and it is precisely equal to $|1_\circ B 1_\circ|$ if B is a left brace.

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