Mixed sign Coxeter Groups

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A Coxeter system (W, S) is a group W with the set of generators

$$S = \{s_1, s_2, ..., s_n\}$$

and relations

$$s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1,$$

where $2 \leqslant m_{ij} \in \mathbb{N}$ or $m_{ij} = \infty$, where the later means that there is no relation between s_i and s_j.

A Coxeter group can also be defined by its Coxeter graph. Its vertex set coincides with the set S of Coxeter generators.

Two vertices s_i and s_j are not connected if $m_{ij} = 2$, are connected by an unlabeled edge if $m_{ij} = 3$ and are connected by an edge labeled by m_{ij} if $m_{ij} > 3$.

Edges labeled by m_{ij} with $m_{ij} > 3$ are called *multiple edges*.

A Coxeter group is called *simply laced* if its Coxeter graph does not have multiple edges, i.e., m_{ij} equals to 2 or 3 for all $i \neq j$.

Any Coxeter group has the canonical linear representation called the *geometric representation* defined in the following way. The dimension of this representation equals to the number of Coxeter generators. For each pair of generators (s_i, s_j), which do not commute, choose a positive number k_{ij} such that

$$k_{ij}k_{ji} = 4 \cdot \cos^2\left(\frac{\pi}{m_{ij}}\right);$$

if $m_{ij} = \infty$, put $k_{ij} = k_{ji} = 2$.

Each generator s_i is mapped to the matrix σ_i which differs from the identity matrix only in its i-th row.

The diagonal element in the position (i,i) is -1, and for $i \neq j$ the entry in the position (i,j) equals to k_{ij};

if the generators s_i and s_j commute, then the entry in the position (i,j) is zero.

Numbers k_{ij} and k_{ji} may be different. If all the numbers m_{ij} are 2,3,4,6 or ∞ it is possible to choose all k_{ij} integers.

The representation $s_i \mapsto \sigma_i$ is called the *geometric representation*.

If for any i, j, $k_{ij} = k_{ji} = 2 \cdot \cos\left(\frac{\pi}{m_{ij}}\right)$, then the representation $s_i \mapsto \sigma_i$ is called the *standard geometric representation*.

When the group is simply laced, all non-zero entries of matrices σ_i of the standard geometric representation are ± 1.

In this talk we deal with a generalization of Coxeter graphs, namely, with mixed-sign Coxeter graphs, i.e., the vertices of the graph are signed by 1 or by -1.

There is an associated representation, which is a generalization of the standard geometric representation of Coxeter groups, which was defined by Hironaka (2013).

The group which we get is called mixed-sign Coxeter group.

The motivation of Hironaka to define mixed-sign Coxeter groups comes from studying the construction of Pseudo-Anosov mapping classes from generalized Coxeter graphs.

Armstrong has showed in his Ph.D. thesis (2013), that every mixed-sign Coxeter group is a quotient of a certain Coxeter group, which graph depends on the signs of the vertices of the corresponding mixed-sign graph.

In this talk we give a description of it in terms of generators and relations, for some important cases.

We say that the triple $\Gamma_f = (V, E, f)$ is a *mixed-sign graph*

$\Gamma = (V, E)$ is a graph,

f is a function from the set of its vertices V to $\{1\,,\,-1\}$.

The mixed-sign Coxeter graph $\Gamma_f = (V, E, f)$:

Let (W, S) be a Coxeter system, where

$S = \{s_1, s_2, ..., s_n\}$,

let Γ be its Coxeter graph, and

let f be a function on vertices of Γ with values ± 1, i.e.,

$$f(\{s_i\}) \in \{1, -1\}.$$

The generator s_i is mapped to the $n \times n$ matrix ω_i which differs from the identity matrix only by the i-th row.

The i-th row of the matrix ω_i has -1 at the position (i, i),

it has $2 \cdot f\left(\{s_j\}\right) \cdot \cos\left(\frac{\pi}{m_{ij}}\right)$ in the position (i, j) when the node s_j is connected to the node s_i, i.e., when $m_{ij} > 2$,

it has 0 in the position (i, j) when the nodes s_j and s_i are not connected by an edge, i.e., when s_j and s_i commute.

In case of the node s_j connected to the node s_i by a simply-laced edge, i.e., when $m_{ij} = 3$, we get

$$2 \cdot f\left(\{s_j\}\right) \cdot \cos\left(\frac{\pi}{3}\right) = f\left(\{s_j\}\right).$$

We denote by W_f the group generated by

$$\omega_1, \omega_2, \ldots, \omega_n.$$

W_f is called mixed-sign Coxeter group.

Consider the generators ω_i of W_f. Then, the following relations are satisfied:

- ${\omega_i}^2 = 1$;

- $\omega_i \cdot \omega_j = \omega_j \cdot \omega_i$,

  if v_i and v_j are not connected by an edge;

- $\omega_i \cdot \omega_j \cdot \omega_i = \omega_j \cdot \omega_i \cdot \omega_j$,

  if v_i and v_j are connected by a simply-laced edge,

  $f(\{s_i\}) \cdot f\left(\{s_j\}\right) = 1$

  (i.e., v_i and v_j have the same signs);

- $(\omega_i \cdot \omega_j)^{m_{ij}} = 1$,

  if v_i and v_j are connected by an edge,

  $f(\{s_i\}) \cdot f\left(\{s_j\}\right) = 1$;

- $\omega_i \cdot \omega_j$ has infinite order in the mixed-sign Coxeter group,

  if v_i and v_j are connected by an edge,

  $f(\{s_i\}) \cdot f\left(\{s_j\}\right) = -1$.

  (i.e., The signs of v_i and v_j are different);

Theorem 1:

Let P be a connected cycle free path in Γ, with the nodes

$v_{i_1}, v_{i_2}, \ldots, v_{i_t}$,

such that the following are satified:

- There exists $1 \leq r \leq t-1$, such that v_{i_r} is connected to $v_{i_{r+1}}$ by a not necessarily simply-laced edge, which is labeled by $m_{i_r i_{r+1}}$;

- v_{i_j} is connected to $v_{i_{j+1}}$ by a simply-laced edge, for every $j \neq r$, such that $1 \leq j \leq t-1$;

- $f\left(\{s_{i_1}\}\right) = f\left(\{s_{i_t}\}\right)$.

Then:

$$\left(\omega_{i_1} \cdot {\omega_{i_t}}^{\omega_{i_{t-1}} \cdot \omega_{i_{t-2}} \cdots \omega_{i_2}}\right)^{m_{i_r i_{r+1}}} = 1$$

In particular, in the simply-laced case:

$$\left(\omega_{i_1}, {\omega_{i_t}}^{\omega_{i_{t-1}} \cdot \omega_{i_{t-2}} \cdots \omega_{i_2}}\right)^3 = 1.$$

Theorem 2:

Let C_t be a cycle in Γ, with the nodes

$v_{i_1}, v_{i_2}, \ldots, v_{i_t}$,

such that v_{i_j} is connected to $v_{i_{j+1}}$ for every $1 \leq j \leq t-1$,

v_{i_t} is connected to v_{i_1}.

The following are satisfied:

- $\frac{f\Big(\{s_{i_1}\}\Big)\cdot f\Big(\{s_{i_2}\}\Big)\cdots f\left(\{s_{i_t}\}\right)}{f\Big(\{s_{i_p}\}\Big)\cdot f\Big(\{s_{i_q}\}\Big)} = -1,$

  where $p \neq q$, and $1 \leq p, q \leq t$;

- There is a one-to-one correspondence between the non-simply-laced edges with each label $m \geq 4$ in the two disjoint sub-paths which connects the vertices p and q in the cycle C_t,

  i.e., For every $r \in \{p, p+1, \ldots, q-1\}$, such that

  $m_{i_r i_{r+1}} \geq 4$, there exists

  $z \in \{q, q+1, \ldots, t, 1, 2, \ldots, p-1\}$, such that

  $m_{i_z i_{z+1}} = m_{i_r i_{r+1}}$.

Then:

$$[\omega_{i_p}{}^{\omega_{i_{p-1}}{}^{\cdots\omega_{i_{p-2}}{}^{\cdots\omega_{i_q+1}}}}, \omega_{i_q}{}^{\omega_{i_{q-1}}{}^{\cdot\omega_{i_{q-2}}{}^{\cdots\omega_{i_p+1}}}}] = 1.$$

Which is equivalent to:

$$[\omega_{i_p}{}^{\omega_{i_p+1}{}^{\cdots\omega_{i_p+2}{}^{\cdots\omega_{i_{q-1}}}}}, \omega_{i_q}{}^{\omega_{i_q+1}{}^{\cdot\omega_{i_q+2}{}^{\cdots\omega_{i_{p-1}}}}}] = 1.$$

Theorem 3:

If the cycle C_t satisfies the following conditions:

- $\frac{f\left(\{s_{i_1}\}\right)\cdot f\left(\{s_{i_2}\}\right)\cdots f\left(\{s_{i_t}\}\right)}{f\left(\{s_{i_p}\}\right)\cdot f\left(\{s_{i_q}\}\right)} = -1$,

  where $p \neq q$, and $1 \leq p, q \leq t$;

- $f\left(\{s_{i_p}\}\right) \cdot f\left(\{s_{i_q}\}\right) = 1$

  (i.e., $f\left(\{s_{i_p}\}\right) = f\left(\{s_{i_q}\}\right)$);

- $|k_{i_p--i_q} - k_{i_q--i_p}| = 2 \cdot \cos\left(\frac{\pi}{m}\right\}$

  for some positive integer $m \geq 3$.

  $k_{i_p--i_q} = \prod_{j=p}^{q-1} k_{i_j i_{j+1}}$.

  $k_{i_q--i_p} = \prod_{j=q}^{p-1} k_{i_j i_{j+1}}$.

Then:

$$\left(\omega_{i_p}{}^{\omega_{i_{p+1}}\cdots\omega_{i_{p+2}}\cdots\omega_{i_{q-1}}}, \omega_{i_q}{}^{\omega_{i_{q+1}}\cdot\omega_{i_{q+2}}\cdots\omega_{i_{p-1}}}\right)^m = 1$$

which is equivalent to

$$\left(\omega_{i_p}{}^{\omega_{i_{p-1}}\cdots\omega_{i_{p-2}}\cdots\omega_{i_{q+1}}}, \omega_{i_q}{}^{\omega_{i_{q-1}}\cdot\omega_{i_{q-2}}\cdots\omega_{i_{p+1}}}\right)^m = 1$$