Universitat de València

Garside Groups Factorizations

Advances in Group Theory, Lecce 2019

Presented by: Raúl Sastriques Guardiola
Directed by: Sergio Camp and Adolfo Ballester

Garside groups

1. Introduction to Garside groups
2. Properties
3. Factorizing Garside groups

Definitions

Cancellative Monoid

- A monoid M is said to be right cancellative if for every $a, b, c \in M$:
$$a \cdot c = b \cdot c \Rightarrow a = b$$
- A monoid M is said to be left cancellative if: for every $a, b, c \in M$:
$$c \cdot a = c \cdot b \Rightarrow a = b$$
- A monoid M is said to be cancellative if it is left and right cancellative.

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Definition (Atom)

An element a in a monoid M is an atom in M if:

$$a = b \cdot c \Rightarrow b = 1 \text{ or } c = 1$$

For any element x in M, $\| x \|$ is the supremum of the lengths of all expressions of x in terms of atoms of M.

Definition (Atomic monoid)

1. M is generated by its atoms.
2. $\| x \| < \infty$, for every $x \in M$.

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Divisibility

If M is a monoid, and $a, b \in M$, we say that a left divides b if there is some element $c \in M$ such that $a \cdot c = b$.
Right divisibility is defined in a similar way.

We may consider two associated orders in the monoid:

$$a \leq_L b \text{ if } a \text{ left divides } b,$$

$$a \leq_R b \text{ if } a \text{ right divides } b.$$

When these two order are lattices, then, for each pair of elements $a, b \in M$, there exist a least common multiple and a greatest common divisor ($a \vee b$ and $a \wedge b$, respectively).

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A monoid M is a Garside monoid when:

1. M is cancellative.
2. M is atomic.
3. $\leq_R$ and $\leq_L$ are both lattices in M.
4. There exists an element $\Delta \in M$, called a Garside element of M, such that:
    4.1 For each $a \in M$, $a \leq_R \Delta$ iff $a \leq_L \Delta$.
    4.2 The set of divisors of Δ is finite and generates M.

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Properties

If M is a Garside monoid, then:

- M is conical, that is, if $a, b \in M$ are so that $a \cdot b = 1$, then $a = 1$ or $b = 1$.
- M is torsion-free.
- All the atoms of M divide its Garside element Δ.
- Δ^n is a Garside element, for every $n \geq 1$.
- M satisfies Ore's conditions. Then it is possible the following definition:

Garside Group

A Garside group is the group of fractions of some Garside monoid M. If G is a Garside group, G^+ denotes its associated monoid.

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Quasi-centre

"Groupes de Garside", 2002

If G is a Garside group with Garside element Δ, and $a \in G^+$, then $a^\Delta \in G^+$.

Then, Δ permutes the atoms of G^+ by conjugation. Since these are finite in number, there exists $n \geq 1$ with:

$$\Delta^n \in Z(G).$$

"The Centre of Thin Gaussian Groups", J. Algebra, 2001

The quasi-centre of a Garside group G is the subgroup:

$$QZ(G) = \{g \in G \mid a^g \in G^+, \text{ for every } a \in G^+\}$$

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The Centre of Thin Gaussian Groups [3]

Theorem (Picantin)

If G is a Garside group, $QZ(G)$ is a finitely generated free abelian group.

A basis for the submonoid generating this subgroup is given.

Definition

A Garside group G is said pure Garside if $QZ(G)$ is cyclic.

In can be shown that G is pure Garside if and only if $Z(G)$ is cyclic.

Theorem (Picantin)

Every Garside monoid is the crossed product of some pure Garside submonoids.

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Zappa-Szép products

Zappa, in 1942, and Szép, in 1950, studied factorizations of a group G as product of a pair of subgroups. In the case of monoids, this product is called 'Zappa-Szép product'.

Definition (Zappa-Szép product)

A monoid M is the (internal) Zappa-Szép product of two submonoids A and B, $M = A \bowtie B$, if every element $x \in M$ can be uniquely written as $x = a \cdot b = b' \cdot a'$, with $a, a' \in A$, $b, b' \in B$.

Like the product of subgroups, the Zappa-Szép product of monoids is commutative, but it is not associative.

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Volker Gebhardt and Stephen Tawn 2015 [2]

Theorem

If M is a Garside monoid, and $M = A \bowtie B$, then A and B are Garside monoids.
Additionally, if Δ is a Garside monoid of M, $\Delta = a \cdot b$ with $a \in A$ and $b \in B$, then a, b are Garside elements of A and B, respectively.

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Indecomposable monoids

Definition (Indecomposable)

A Garside monoid M is indecomposable if it cannot be written as a Zappa-Szép product of two non-trivial submonoids.

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A Garside monoid is pure Garside if and only if it is indecomposable.

Since the are a finitely many atoms, the number of pure Garside submonoids when recursively factorizing a Garside monoid as the Zappa-Szép product of two submonoids is also finite.

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Pure factors

Any Garside monoid G^+ can be factorized as a recursive Zappa-Szép product of pure factors.

Uniqueness

If G is a Garside group and there are two factorizations $G^+ = H_1^+ \bowtie \ldots \bowtie H_n^+ = K_1^+ \bowtie \ldots \bowtie K_m^+$, (parentheses omitted) where the products of the submonoids are Zappa-Szép, and H_i^+, K_j^+ are pure Garside groups, then $n = m$ and, for each $i = 1, \ldots n$, there is $j \in \{1, \ldots, n\}$ such that $H_i^+ = K_j^+$.

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Minimal Garside element

In [1], P. Dehornoy proved the existence of a (unique) minimal Garside element. Regarding to the factorization of G, we have the following result.

Minimal Garside element

Let G be a Garside group and let δ_i be the generators of the quasi-centre of the pure factors of $G^+ = H_1^+ \bowtie H_2^+ \bowtie \ldots \bowtie H_n^+$. Then $\Delta = \delta_1 \cdot \delta_2 \cdot \ldots \cdot \delta_n$ is the minimal Garside element of G (with respect to $\leq_R$ and $\leq_L$).

For every $t > 0$,

$$\Delta^t = \delta_1^t \cdot \delta_2^t \cdot \ldots \cdot \delta_n^t$$

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Principal factors

Definition

If $x_1, \ldots, x_r$ is a basis of the submonoid $QZ(G)^+$, then we may define N_j as the subgroup generated by the atoms dividing the element x_j.

Principal factorization

If G^+ is a Garside monoid, then

$$G^+ = N_1^+ \bowtie N_2^+ \bowtie \cdots \bowtie N_r^+.$$

In particular, the product of the N_j is pairwise permutable.

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Garside elements

Characterization

If Δ_j be the minimal Garside element of the principal factor N_j. Then Δ is a Garside element of G if and only if $\Delta = \Delta_1^{t_1} \cdot \ldots \cdot \Delta_r^{t_r}$, for some $t_1 \ldots t_r > 0$.

Isomorphic pure factors

If N_j is a principal factor of G, then the pure factors of N_j are all isomorphic.

Proposition

A Garside group is abelian if and only if its principal factors are all cyclic.

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Example

The Garside monoid
$G^+ = \left\{a, b, c, d \mid ab = ba, ac = ca, bc = cb, a^d = a, b^d = c, c^d = b\right\}$,
admits two different factorizations

$$G^+ = \left(\left(A^+ \times B^+\right) \times C^+\right) \rtimes D^+, \tag{1}$$

where the action is given by $a^d = a$, $b^d = c$, $c^d = b$, and

$$G^+ = \left(B^+ \times C^+\right) \rtimes \left(A^+ \times D^+\right), \tag{2}$$

with action $b^a = b$, $c^a = c$, $b^d = c$, $c^d = b$.
In particular, we see that the pure factors A^+, B^+, C^+, D^+ are the same for both factorizations, and also their length.

Example

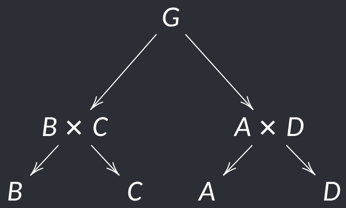

Figure: Decomposition (2)

$\Delta = b \cdot c \cdot a \cdot d$ is the minimal Garside element of G^+, and $(bc), (ad)$ are quasi-central in G^+. Moreover, conditions $bd = dc, cd = db$ imply that b, c cannot be quasi-central while $d \in QZ(G)^+$ since $ad = da$. In particular

$$G^+ = \langle b, c \rangle \bowtie \langle a \rangle \bowtie \langle d \rangle, \qquad QZ(G)^+ = \langle bc, a, d \rangle,$$

and the set of all the Garside elements of G is:

$$\left\{ (bc)^{e_1} a^{e_2} d^{e_3} \mid e_i \in \mathbb{N}, \text{ and } e_i \geq 1 \text{ for all } i \right\}.$$

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Grazie Mille !

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