Metahomomorphisms on groups and the Yang-Baxter equation

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Definition

A **set-theoretic solution** to the Yang-Baxter equation (YBE) is a pair (X, r), where X is a non-empty set and $r : X \times X \to X \times X$, $(x, y) \mapsto (\lambda_x(y), \rho_y(x))$ is a map such that

$$(r \times id)(id \times r)(r \times id) = (id \times r)(r \times id)(id \times r).$$

If λ_x [resp. ρ_x] is bijective, for every $x \in X$, the solution (X, r) is said to be *left* [resp. *right*] *non-degenerate*.

Examples

- The *flip* map $r(x, y) = (y, x)$;
- (Lyubaschenko) If f, g are maps $X \to X$, then

$$r(x, y) = (f(y), g(x)),$$

for all $x, y \in X$, is a solution iff $fg = gf$;

Examples

- (Venkov) Let X be a set , $\circ$ be an operation on X. Then

$$r(x, y) = (x \circ y, x),$$

for all $x, y \in X$, is a solution iff

$$x \circ (y \circ z) = (x \circ y) \circ (x \circ z);$$

- (Gu) Let G be a group , τ be a map from G to itself. Then

$$r(x, y) = (xy\tau(x)^{-1}, \tau(x))$$

for all $x, y \in G$, is a solution iff

$$\tau(xy\tau(x)^{-1}) = \tau(x)\tau(y)\tau^2(x)^{-1}.$$

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Let G be a group and τ be a map from G to itself. If τ satisfies

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Some examples:

- All the endomorphisms of G;
- The map $\tau(x) = x^{-1}$;
- The map $k_g(x) = g$.

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Definition

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Proposition (Gu, '98)

If τ is a metahomomorphism on a group G, then

$$\phi(x) := \tau(x)\tau(1)^{-1}$$

is a unitary metahomomorphism on G.

For example: $\phi(x) := k_g(x)k_g(1)^{-1}$ is a unitary metahomomorphism

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Theorem (Gu, '97)

Let G be a finite simple group, τ be a unitary metahomomorphism on G. Then, τ must be a homomorphism or τ satisfies $\tau(x) = x^{-1}$, for every $x \in G$.

What can we say about abelian groups?
Let G be an abelian group. Just for convenience, we denote the operation on G by $+$ and the unit in G by 0. Therefore a map τ on G is a metahomomorphism iff

$$\tau(x + y - \tau(x)) = \tau(x) + \tau(y) - \tau^2(x),$$

for all $x, y \in X$.

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Proposition (Ding, GU, 2006)

Let G be an abelian group and $\tau : G \to G$. Then, the following statements are equivalent:

(i) τ *is a unitary metahomomorphism;*

(ii) *there exist a subgroup H of G, a representative coset X of G/H containing 0, a function $f : X \to H$ such that $f(0) = 0$ and $\alpha \in End(H)$ such that $\tau(x + h) = x + f(x) + \alpha(h)$, for all $x \in X, h \in H$.*

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Let G be an abelian group and τ a unitary metahomomorphism on G, then (X, r), where $r : G \times G \to G \times G$, is a set-theoretic solution to the YBE if

$$r(x, y) = (x + y - \tau(x), \tau(x)).$$

Definition

A set-theoretic solution (X, r) is said to be *involutive* if if

$$r^2 = id_{X \times X}.$$

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If G is an abelian group and τ a unitary metahomomorphism on G, then

$$r(x,y) = (x + y - \tau(x), \tau(x)),$$

is an involutive non-degenerate set-theoretic solution iff

$$x + y - \tau(x) = \tau^{-1}(y).$$

This is an already known solution (Lyubaschenko's solution).

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Using unitary metahomomorphism on abelian groups is it possible to define, in a different way, other involutive set-theoretic solutions?

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Definition (Rump, 2005)

Let X be a non-empty set and $\cdot$ a binary operation on X. The pair $(X, \cdot)$ is said to be a *cycle set* if each left multiplication $\sigma_x(y) := x \cdot y$ is bijective, for all $x, y \in X$ and the following

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z),$$

for all $x, y, z \in X$.

(Rump) There exists a bijective correspondence between cycle set and left non-degenerate involutive set-theoretic solutions.

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Definition (Rump, 2016)

A triple $(G, +, \cdot)$ is said to be a *quasi-linear cycle set* if $(G, \cdot)$ is a cycle sets, $(G, +)$ is an abelian group and

$$x\cdot (y + z) = x\cdot y + (x - y)\cdot z$$

holds for all $x, y, z \in G$. The permutation τ given by $\tau(x) := 0\cdot x$ for every $x \in G$ is called the *associated permutation* to G.

For example:
Let G be an abelian group, then any automorphism τ of G makes G into a quasi-linear left cycle set with $x \cdot y = \tau(y)$, for all $x, y \in G$.

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Proposition (Rump, 2016)

Let $(G, +, \cdot)$ be a quasi-linear cycle set and τ its associated permutation. Then $\tau(0) = 0$ and

$$\sigma_x(y) = \tau(y - x) - \tau(-y)$$

for all $x, y \in X$.

Conversely

Proposition (Rump, 2016)

Let G be an abelian group and let τ be a permutation of G with $\tau(0) = 0$. Define

$$\sigma_x(y) := \tau(y - x) - \tau(-y),$$

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Definition (Rump, 2016)

If $(G, \cdot, +)$ is a quasi-linear cycle sets we can define two standard subgroups:
- the *Socle* of G is the set

$$Soc(G) := \{x | x \in G \; \forall y \in G \; x \cdot y = 0 \cdot y\};$$

- the *Radical* of G is the subgroup generated by the elements

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Theorem (Castelli, Catino, Miccoli, P., 2018)

Let $(G, +, \cdot)$ be a quasi-linear cycle set and τ be its associated permutation. Then, the following statements are equivalent:

(i) $Rad(G) \subseteq Soc(G)$;

(ii) τ *is a unitary metahomomorphism.*

A unitary metahomomorphism of an abelian group is not necessarily the associated permutation of a quasi-linear cycle set.

Proposition (Castelli, Catino, Miccoli, P., 2018)

If τ is a unitary metahomomorphism of an abelian group G such that $Rad(\tau) \subseteq Fix(\tau)$, then τ is the associated permutation to a quasi-linear cycle set with G as underlying additive group and such that $Rad(G) \subseteq Fix(G)$.

Where $Rad(\tau)$ is the subgroup generated by the elements $\tau(x) - x$ and the $Fix(G) := Soc(G) \cap Fix(\tau)$.

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If a unitary metahomomorphism τ is the associated permutation of a quasi-linear cycle set, we know that

$$\sigma_x(y) = \tau(y - x) - \tau(-y).$$

(Rump) Let $r : G \times G \to G \times G$ be the map given by

$$r(x, y) = (\sigma_x^{-1}(y), \sigma_{\sigma_x^{-1}(y)}(x)),$$

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Thanks for the attention!