Equivariant IYB-structures in finite groups[1]

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[1] Joint work with Adolfo Ballester-Bolinches and Ramón Esteban-Romero

- F. Cedó, E. Jespers and Á. Del Río. "Involutive Yang-Baxter groups." Transactions of the American Mathematical Society 362.5 (2010): 2541-2558.

Definition

Let X be a non-empty set and $r : X \times X \to X \times X$ is a map such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where the maps r_{12}, r_{23}: $X \times X \times X \to X \times X \times X$ are defined as $r_{12} = r \times id_X$, $r_{23} = id_X \times r$. Then (X, r) is called a set-theoretic solution of the Yang-Baxter equation.

Let (X, r) be a set-theoretic solution of the Yang-Baxter equation. Set $r(x, y) = (f_x(y), g_y(x))$, where $f_x, g_y : X \to X$ are two maps, $x, y \in X$. Then (X, r) is called

- involutive if $r^2 = id_{X\times X}$;
- non-degenerate if f_x, g_y are bijective maps for all $x, y \in X$.

Let (X, r) be an involutive, non-degenerate set-theoretic solution of the Yang-Baxter equation and set $r(x, y) = (f_x(y), g_y(x))$ for all $x, y \in X$. The permutation group of (X, r) is defined as

$$\mathcal{G}(X, r) = \langle f_x : x \in X \rangle \leq S_X.$$

Definition

A group G is called an involutive Yang-Baxter group (for short, IYB-group) if $G \cong \mathcal{G}(X, r)$ for (X, r) an involutive, non-degenerate set-theoretic solution of the Yang-Baxter equation.

Theorem

Every finite IYB-group is soluble.

But the converse is not ture.

Theorem(D. Bachiller, 2016)

There exists a finite nilpotent (p-)group such that it is not an IYB-group.

- D. Bachiller, "Counterexample to a conjecture about braces." Journal of Algebra 453 (2016): 160-176.

Question

Classify or characterize finite IYB-groups.

Equivalent Statements

The following conditions are equivalent for a finite group G.

- G is an IYB-group;
- There is a G-module V and a bijective 1-cocycle $\pi : G \to V$.

- (V, π) is called an IYB-structure on G;
- 1-cocycle π of G-module V is a map from G to V such that $\pi(gh) = \pi(g) + g\pi(h)$ for any $g, h \in G$.

Some Known Results

Let G be a finite group. Then G is an IYB-group if one of the following conditions holds.

- G is nilpotent of class at most two;
- G is abelian-by-cyclic;
- G is a soluble A-group (that is, every Sylow subgroup is abelian);
- G is the direct product of two IYB-groups.

Question

What about the semidirect product of two IYB-groups?

In order to consider the semidirect product of two IYB-groups, F. Eisele introduce

Definition

Let (V, π) be an IYB-structure on an IYB-group G and let a group A act on G. (V, π) is called A-equivariant if there exists an group action of A on V (denote by $av, a \in A, v \in V$) such that

$$\pi({}^a g) = a\pi(g), \forall a \in A, g \in G.$$

In fact, since π is bijective, such action of A on V is uniquely determined by the action of A on G as follows:

$$A \times V \to V; (a, v) \to av \triangleq \pi({}^a \pi^{-1}(v)).$$

Proposition

Let (V, π) be an IYB-structure on a group G and let a group A act on G. Write $K = \mathrm{Ker}(A\ on\ G)$, the kernel of the action of A on G. Then the following statements are equivalent:

- (V, π) is an A-equivariant IYB-structure on G;
- (V, π) is an A/K-equivariant IYB-structure on G.

- Only focus on the faithful action of A on G.
- An IYB-structure (V, π) on a group G is called *fully equivariant* if it is $Aut(G)$-equivariant.

Example

Let G be a finite abelian group. Let $V = (G, +)$ be a trivial G-module. Obviously (V, Id_G) is fully equivariant and $G = \mathrm{Ker}(G \text{ } on \text{ } V)$.

Example

Let G be an odd order nilpotent group of class at most two. Then G is an abelian group under the following addition:

$$g_1 + g_2 \triangleq g_1 g_2 \sqrt{[g_2, g_1]},$$

and $V = (G, +)$ can be regarded as a G-module:

$${}^{g}v \triangleq gv + g^{-1}.$$

Note that (V, Id_G) is fully equivariant and $\mathrm{Z}(G) = \mathrm{Ker}(G \text{ } on \text{ } V)$.

Example(J. C. Ault and J. F. Watters, 1973)

Let G be a finite nilpotent group of class at most two. Set $Z = \mathrm{Z}(G)$ and write $G/Z = \langle a_1 Z \rangle \times \dots \times \langle a_n Z \rangle$. Thus every element of G can be written in this form: $a_1^{t_1} \dots a_n^{t_n} z$, where $z \in Z$. Then G is an abelian group under the following addition:

$$a_1^{t_1} \dots a_n^{t_n} z + a_1^{s_1} \dots a_n^{s_n} z' = a_1^{t_1+s_1} \dots a_n^{t_n+s_n} zz'.$$

Let $V = (G, +)$ be a G-module by law:

$$^{g}v \triangleq gv - g = v \prod_{1 \le j < i \le n} [a_i, a_j]^{t_i s_j},$$

where $g = a_1^{t_1} \dots a_n^{t_n} z \in G$ and $v = a_1^{s_1} \dots a_n^{s_n} z' \in V$. Then (V, Id_G) is an IYB-structure on G and $\mathrm{Z}(G) \subseteq \mathrm{Ker}(G \ on \ V)$.

Proposition

Let G be a nilpotent group of class at most two. Then G has an IYB-structure (V, π) such that

- (V, π) is $\mathrm{Aut}_c(G)$-equivariant;
- $\mathrm{Z}(G) \subseteq \mathrm{Ker}(G \textit{ on } V)$.

- A central automorphism α of G: ${}^{\alpha}gg^{-1} \in \mathrm{Z}(G)$ for all $g \in G$.
- $\mathrm{Aut}_c(G)$: the set of all central automorphisms of G.

Proposition(F. Cedó, E. Jespers and Á. Del Río, 2010;F. Eisele, 2013)

Let a finite group $G = [N]H$ be the semidirect product of two IYB-groups N and H. If N has an H-equivariant IYB-structure, then G is an IYB-group.

- F. Cedó, E. Jespers and Á. Del Río. "Involutive Yang-Baxter groups." Transactions of the American Mathematical Society 362.5 (2010): 2541-2558.
- F. Eisele. "On the IYB-property in some solvable groups." Archiv der Mathematik 101.4 (2013): 309-318.

For general products, an interesting result is the following.

Theorem(F. Cedó, E. Jespers and Á. Del Río, 2010)

Let G be a finite group such that $G = NH$, where N is an abelian normal subgroup of G and H is a subgroup of G with an IYB-structure (V, π). Suppose that

$$N \cap H \subseteq \mathrm{Ker}(H \ on \ V).$$

Then G is an IYB-group.

Main Theorem

Let a group A act on a group $G = NH$ such that N, H are A-invariant subgroups of G and $N \trianglelefteq G$. Suppose that N, H have A-equivariant IYB-structures (U, π_N) and (V, π_H) respectively, which satisfy the following conditions:

(C1) $N \cap H \subseteq \mathrm{Ker}(\mathrm{Z}(N) \ on \ U) \cap \mathrm{Ker}(H \ on \ V)$;

(C2) (U, π_N) is H-equivariant under the conjugacy action of H on N: ${}^h n = hnh^{-1}, n \in N, h \in H$.

Then G has an A-equivariant IYB-structure (W, π) such that

$$\mathrm{Ker}(N \ on \ U)\, \mathrm{C}_{\mathrm{Ker}(H \ on \ V)}(N) \subseteq \mathrm{Ker}(G \ on \ W).$$

Corollary

Let a group A act on a group $G = N \times H$, the direct product of two A-invariant subgroups N and H. Suppose that N, H are IYB-groups with A-equivariant IYB-structures (U, π_N) and (V, π_H) respectively. Then G has an A-equivariant IYB-structures (W, π_G) such that

$$\mathrm{Ker}(N \textit{ on } U)\,\mathrm{Ker}(H \textit{ on } V) \subseteq \mathrm{Ker}(G \textit{ on } W).$$

With this, we can prove

Corollary

Let G be a nilpotent group of class two with an abelian Sylow 2-subgroup. Then G has a fully equivariant IYB-structure (W, π_G) such that $\mathrm{Z}(G) \subseteq \mathrm{Ker}(G \textit{ on } W)$.

Corollary

Let a finite group $G = NH$ such that N is a nilpotent normal subgroup of class at most two and H is a subgroup of G with IYB-structure (V, π). Assume that

- $N \cap H \subseteq \mathrm{Z}(N)$;
- $[H, \mathrm{O}_2(N)] \subseteq \mathrm{Z}(N)$;
- $H \cap N \subseteq \mathrm{Ker}(H \text{ on } V)$.

Then G is an IYB-group.

- This is an extension of the case that N is abelian.

Corollary

Let a finite group $G = NH$ such that N, H are two nilpotent subgroups of class at most two and N is normal in G. If $N \cap H \subseteq \mathrm{Z}(G)$ and $[H, \mathrm{O}_2(N)] \subseteq \mathrm{Z}(N)$. Then G is an IYB-group.

The following corollary is a special case,

Corollary(F. Cedó, E. Jespers and Á. Del Río, 2010)

Let G be a finite group. If $G = NH$, where N and H are two abelian subgroups of G and N is normal in G, then G is an IYB-group.

Corollary

Let a finite group $G = N_1 N_2 ... N_s$ the product of subgroups $N_1, ..., N_s$. Suppose that

- N_i is a nilpotent group of class two with an abelian Sylow 2-subgroup, $i = 1, ..., s$;
- N_i is normalised by N_j, for all $1 \leq i < j \leq s$;
- $N_1 ... N_i \cap N_{i+1} = Z(G)$, $i = 1, ..., s-1$.

Then G is an IYB-group.

Thanks for your attention!.