Wide simple groups and Lie algebras

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Further examples and results can be found in a survey paper of Kappe and Morse (2007).

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The cases of finite and infinite groups should be considered separately.
In the case where G is finite, each element is a single commutator. This was conjectured by Ore in the 1950's. The proof required lots of various techniques. Most groups of Lie type were treated by Ellers and Gordeev in the 1990's. The proof was finished by Liebeck, O'Brien, Shalev and Tiep in 2010. See Malle's Bourbaki talk (2013) for details.

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- $G = \mathcal{G}(k)$, the group of k-points of a semisimple adjoint linear algebraic group $\mathcal{G}$ over an algebraically closed field k (Ree, 1964);
- G is the automorphism group of some nice topological or combinatorial object (e.g., the Cantor set).

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These groups are indeed very different from “nice” groups discussed above in the following sense.

Commutator width

For any group G one can introduce the following notions.
For any $a \in [G, G]$ define its length $\ell(a)$ as the smallest number k of commutators needed to represent it as a product

$$a = [x_1, y_1] \dots [x_k, y_k].$$

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It turns out that for a simple group G the commutator width $\mathrm{wd}(G)$ may be as large as we wish, or even infinite (such examples appear in the papers of Barge–Ghys and Muranov).

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As in the case of groups, wide Lie algebras naturally appear among finite-dimensional *nilpotent* Lie algebras and also *perfect* Lie algebras (Cornulier).

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More generally, as in the case of groups, one can define for every $a \in [L, L]$ its bracket length $\ell(a)$ as the smallest k such that a is representable as a sum

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then define the bracket width of L as

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If Question (i) is answered in the affirmative, one can ask the next question:
(ii) Does there exist a simple Lie algebra L of infinite bracket width?

Where to look for counter-examples?

Throughout below L is a *simple* Lie algebra over a field k.
First suppose that L is *finite-dimensional*.
In the following cases it is known that every element is a single bracket (i.e., $\mathrm{wd}(L) = 1$):

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- some non-compact algebras L over $\mathbb{R}$ (Akhiezer).

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Working hypothesis. None of these algebras are wide.

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There are several natural families of simple infinite-dimensional Lie algebras. Here are some of them:

- four families W_n, H_n, S_n, K_n of algebras of Cartan type;
- (subquotients of) Kac–Moody algebras;
- algebras of vector fields on smooth affine varieties.

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Observation (due to Zhihua Chang):
A theorem of Rudakov (1969) shows that none of the algebras L of Cartan type are wide.

Back to the origins

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Sophus Lie
(1842–1899)

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Élie Cartan
(1869–1951)

Main result

Among Lie algebras of vector fields on smooth affine varieties there are wide algebras
(B.K. and Andriy Regeta, work in progress).

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$$[\xi, \eta] := \xi \circ \eta - \eta \circ \xi.$$

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- two normal affine varieties are isomorphic if and only if $\mathrm{Vec}(X)$ and $\mathrm{Vec}(Y)$ are isomorphic as Lie algebras (Janusz Grabowski (1978) for smooth varieties, Thomas Siebert (1996) in general);
- X is smooth if and only if $\mathrm{Vec}(X)$ is simple (David Alan Jordan (1986), Siebert (1996); see also Kraft's notes (2017) and a new proof due to Billig and Futorny (2017)).

Example

$X = \mathbb{A}^n$.
$\mathrm{Vec}(\mathbb{A}^n)$ is a free $\mathcal{O}(\mathbb{A}^n) = k[x_1, \ldots, x_n]$-module of rank n generated by $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \ldots, n$.

Main example (Billig–Futorny, 2017)

Let $H = \{y^2 = 2h(x)\}$ where $h(x)$ is a separable monic polynomial of odd degree $2m + 1 \geq 3$,
$A = \mathcal{O}(H) = k[x, y]/\left\langle y^2 - 2h(x)\right\rangle$. As a vector space,
$A \cong k[x] \oplus yk[x]$.
$\mathrm{Vec}(H) = \mathrm{Der}_k(A)$.
Lemma (Billig–Futorny). Vec(H) *is a free* A*-module of rank 1 generated by*

$$\tau = y\partial_x + h'(x)\partial_y.$$

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(We say that η is semisimple if $\mathrm{ad}(\eta)$ has an eigenvector.)

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Idea of proof. One can introduce a filtration on D so that the smallest nonzero degree is $2m - 1$. Then any $\eta \in D$ with $\deg \eta = 2m - 1$ is not representable as a single Lie bracket.

Another example

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Theorem. *Let $\eta = p'(z)\partial_y + x\partial_z$. Then $\eta \in L$ and there are no $\xi, \nu \in L$ such that $[\xi, \nu] = \eta$.*

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Lemma. (Leuenberger–Regeta, 2017). *L is a simple Lie algebra.*
Theorem. *Let $\eta = p'(z)\partial_y + x\partial_z$. Then $\eta \in L$ and there are no $\xi, \nu \in L$ such that $[\xi, \nu] = \eta$.*
The proof is based on the same paper by Leuenberger and Regeta and uses degree arguments.

Bracket width

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Remark. If L is finite-dimensional over any infinite field of characteristic different from 2 and 3, its bracket width is at most two (Bergman–Nahlus, 2011).

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- Where should one look for further examples of wide simple Lie algebras? There are two candidates, both suggested by Yu. Billig. a) 'Kac–Moody' algebras arising from the 'Cartan' matrix $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$; b) algebras of Krichever–Novikov type.

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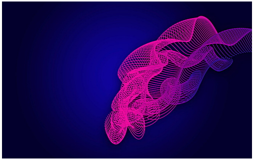

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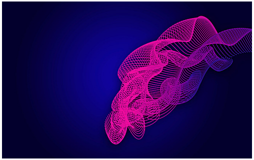

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