The matched product of shelves

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Università del Salento

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The main results of this talk are contained in

F. Catino, **I. Colazzo**, P. Stefanelli, *The matched product of self-distributive systems*, in preparation.

Solutions of the Yang-Baxter equation

The Yang-Baxter equation is a fundamental tool in many fields such as:

- statistical mechanics,
- quantum group theory,
- low-dimensional topology.

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Given X a set, a map $r : X \times X \to X \times X$ is a set-theoretical solution if

$$(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)(r \times \mathrm{id}_X) = (\mathrm{id}_X \times r)(r \times \mathrm{id}_X)(\mathrm{id}_X \times r)$$

Reidemeister move of type III

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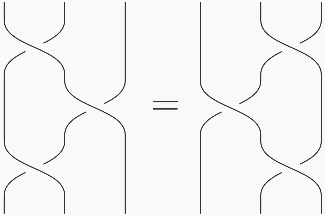

Reidemeister move of type III

Solutions of the Yang-Baxter equation

If X is a set, $r : X \times X \to X \times X$ is a solution and $a, b \in X$, then we denote

$$r(a, b) = (\lambda_a(b), \rho_b(a)),$$

where λ_a, ρ_b are maps from X into itself.

We say that r is

- **left** (resp. right) **non-degenerate** if λ_a (resp. ρ_a) is bijective, for every $a \in X$;
- **idempotent** $r^2(a, b) = r(a, b)$, for all $a, b \in X$
- **involutive** if $r^2(a, b) = (a, b)$, for all $a, b \in X$.

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  Etingof, Schedler and Soloviev; Gateva-Ivanova and Van den Bergh

- 2000. bijective not necessarily involutive solutions
  Lu, Yan and Zhu; Soloviev

- 2017. idempotent solutions
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  free and free commutative monoids, factorizable monoids, distributive lattices, and plactic monoids.

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The shelves

A set X with an operation $\triangleright$ is a left shelf if $\triangleright$ is a left self-distributive operation, i.e.,

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$$

for all $x, y, z \in X$. Moreover, if the maps $L_x : X \to X$ defined by $L_x(y) := x \triangleright y$, for all $x, y \in X$, are bijections, $(X, \triangleright)$ is said to be a left rack.

A set X with an operation $\triangleleft$ is a right shelf if $\triangleleft$ is a right self-distributive operation, i.e.,

$$(x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$$

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Shelves and solutions

From a shelf to a solution of the Yang-Baxter equation...

$$(X,\triangleright) \text{ left shelf} \iff r_\triangleright : X\times X \to X\times X, (x,y)\mapsto (y, y\triangleright x) \text{ solution}$$
$$(X,\triangleleft) \text{ right shelf} \iff r_\triangleleft : X\times X \to X\times X, (x,y)\mapsto (y\triangleleft x, x) \text{ solution}$$

Moreover

$$(X,\triangleright) \text{ left rack} \iff r_\triangleright \text{ non-degenerate solution}$$
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... from a solution of the Yang-Baxter equation to a shelf

If r is a left non-degenerate solution on X, then the binary operation $\triangleright_r$ defined by

$$a \triangleright_r b = \lambda_a \rho_{\lambda_b^{-1}(a)}(b)$$

gives to X a structure of a shelf called the structure shelf.

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The matched product of solutions

*F. Catino, **I.C.**, P. Stefanelli,* The matched product of solutions, *in press on J. Pure Appl. Algebra*

- r_S a solution on a set S
- $\alpha : T \to \mathrm{Sym}(S)$ a map
- r_T a solution on a set T
- $\beta : S \to \mathrm{Sym}(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a **matched product system of solutions** if and only if

$$\alpha_u \alpha_v = \alpha_{\lambda_u(v)} \alpha_{\rho_v(u)} \quad \text{(s1)} \qquad \beta_a \beta_b = \beta_{\lambda_a(b)} \beta_{\rho_b(a)} \quad \text{(s2)}$$

$$\rho_{\alpha_u^{-1}(b)} \alpha^{-1}_{\beta_a(u)}(a) = \alpha^{-1}_{\beta_{\rho_b(a)} \beta_b^{-1}(u)} \rho_b(a) \quad \text{(s3)} \qquad \rho_{\beta_a^{-1}(v)} \beta^{-1}_{\alpha_u(a)}(u) = \beta^{-1}_{\alpha_{\rho_v(u)} \alpha_v^{-1}(a)} \rho_v(u) \quad \text{(s4)}$$

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and the map $r : S\times T \times S\times T \to S\times T \times S\times T$ defined by

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- $\beta : S \to \operatorname{Sym}(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a **matched product system of solutions** if and only if

$$\alpha_u\alpha_v = \alpha_{\lambda_u(v)}\alpha_{\rho_v(u)} \quad \text{(s1)} \qquad \beta_a\beta_b = \beta_{\lambda_a(b)}\beta_{\rho_b(a)} \quad \text{(s2)}$$

$$\rho_{\alpha_u^{-1}(b)}\alpha^{-1}_{\beta_a(u)}(a) = \alpha^{-1}_{\beta_{\rho_b(a)}\beta_b^{-1}(u)}\rho_b(a) \quad \text{(s3)} \qquad \rho_{\beta_a^{-1}(v)}\beta^{-1}_{\alpha_u(a)}(u) = \beta^{-1}_{\alpha_{\rho_v(u)}\alpha_v^{-1}(a)}\rho_v(u) \quad \text{(s4)}$$

$$\lambda_a\alpha_{\beta_a^{-1}(u)} = \alpha_u\lambda_{\alpha_u^{-1}(a)} \quad \text{(s5)} \qquad \lambda_u\beta_{\alpha_u^{-1}(a)} = \beta_a\lambda_{\beta_a^{-1}(u)} \quad \text{(s6)}$$

and the map $r : S\times T \times S\times T \to S\times T \times S\times T$ defined by

$$r\left((a,u),(b,v)\right) := \left(\left(\underbrace{\alpha_u\lambda_{\bar a}(b)}_{=A},\ \underbrace{\beta_a\lambda_{\bar u}(v)}_{=U}\right),\ \left(\alpha_{U}^{-1}\rho_{\alpha_{\bar u}(b)}(a),\ \beta_{A}^{-1}\rho_{\beta_{\bar a}(v)}(u)\right)\right)$$

is a solution, where we denote $\alpha_u^{-1}(a)$ with $\bar a$ and $\beta_a^{-1}(u)$ with $\bar u$, for any (a,u).

The matched product of solutions

F. Catino, ***I.C.***, *P. Stefanelli*, The matched product of solutions, *in press on J. Pure Appl. Algebra*

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The matched product of two left shelves

- $(S, \triangleright)$ is a left shelf
- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}\,(S)$ a map
- $(T, \triangleright)$ is a left shelf
- r_T solution associated with $(T, \triangleright)$
- $\beta : S \to \mathrm{Sym}\,(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions if and only if α_u and β_a are homomorphisms of left shelves

$$\alpha_{v \triangleright u} = \alpha_v^{-1} \alpha_u \alpha_v \qquad \text{(l1)} \qquad\qquad \beta_{b \triangleright a} = \beta_b^{-1} \beta_a \beta_b \qquad \text{(l2)}$$

$$\alpha_u = \alpha_{\beta_a^{-1}(u)} \qquad \text{(l3)} \qquad\qquad \beta_a = \beta_{\alpha_u^{-1}(a)} \qquad \text{(l4)}$$

hold for all $a, b \in S$, $u, v \in T$.

The matched product solution $r_S \bowtie r_T$ is given by

$$r_S \bowtie r_T\left(\left(a, u\right), \left(b, v\right)\right) = \left(\left(\alpha_u(b), \beta_a(v)\right), \left(\alpha_v^{-1}\left(\alpha_u(b) \triangleright a\right), \beta_b^{-1}\left(\beta_a(v) \triangleright u\right)\right)\right)$$

and $r_S \bowtie r_T$ is left non-degenerate.

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and $r_S \bowtie r_T$ is left non-degenerate.

The matched product of two right shelves

- $(S, \triangleleft)$ is a right shelf
- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}\,(S)$ a map

- $(T, \triangleleft)$ is a right shelf
- r_T solution associated with $(T, \triangleright)$
- $\beta : S \to \mathrm{Sym}\,(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions if and only if α_u and β_a are homomorphisms of right shelves

$$\alpha_{u \triangleleft v} = \alpha_u \alpha_v \alpha_u^{-1} \quad \text{(r1)} \qquad \beta_{a \triangleleft b} = \beta_a \beta_b \beta_a^{-1} \quad \text{(r2)}$$

$$\alpha_u = \alpha_{\beta_a^{-1}(u)} \quad \text{(r3)} \qquad \beta_a = \beta_{\alpha_u^{-1}(a)} \quad \text{(r4)}$$

hold for all $a, b \in S$, $u, v \in T$.

The matched product solution $r_S \bowtie r_T$ is given by

$$r_S \bowtie r_T\left((a, u), (b, v)\right) = \left((\alpha_u(b) \triangleleft a, \beta_a(v) \triangleleft u), (\alpha^{-1}_{v \triangleright u}(a), \beta^{-1}_{b \triangleleft a}(u))\right),$$

and $r_S \bowtie r_T$ iis left non-degenerate if and only if $(S, \triangleleft)$ and $(T, \triangleleft)$ are right racks.

The matched product of two right shelves

- $(S, \triangleleft)$ is a right shelf
- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}\,(S)$ a map
- $(T, \triangleleft)$ is a right shelf
- r_T solution associated with $(T, \triangleright)$
- $\beta : S \to \mathrm{Sym}\,(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions if and only if α_u and β_a are homomorphisms of right shelves

$$\alpha_{u \triangleleft v} = \alpha_u \alpha_v \alpha_u^{-1} \qquad \text{(r1)} \qquad\qquad \beta_{a \triangleleft b} = \beta_a \beta_b \beta_a^{-1} \qquad \text{(r2)}$$

$$\alpha_u = \alpha_{\beta_a^{-1}(u)} \qquad \text{(r3)} \qquad\qquad \beta_a = \beta_{\alpha_u^{-1}(a)} \qquad \text{(r4)}$$

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The matched product solution $r_S \bowtie r_T$ is given by

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- $(S, \triangleleft)$ is a right shelf
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$(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions if and only if α_u and β_a are homomorphisms of right shelves

$$\alpha_{u \triangleleft v} = \alpha_u \alpha_v \alpha_u^{-1} \quad \text{(r1)} \qquad \beta_{a \triangleleft b} = \beta_a \beta_b \beta_a^{-1} \quad \text{(r2)}$$

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and $r_S \bowtie r_T$ iis left non-degenerate if and only if $(S, \triangleleft)$ and $(T, \triangleleft)$ are right racks.

The matched product of left and right shelves

- $(S, \triangleright)$ is a left shelf
- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}(S)$ a map

- $(T, \triangleleft)$ is a right shelf
- r_T solution associated with $(T, \triangleright)$
- $\beta : S \to \mathrm{Sym}(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions if and only if α_u are homomorphisms of left shelves and β_a are homomorphisms of right shelves

$$\alpha_{u \triangleleft v} = \alpha_u \alpha_v \alpha_u^{-1} \quad \text{(lr1)} \qquad \beta_{a \triangleright b} = \beta_b^{-1} \beta_a \beta_b \quad \text{(lr2)}$$

$$\alpha_u = \alpha_{\beta_a^{-1}(u)} \quad \text{(lr3)} \qquad \beta_a = \beta_{\alpha_u^{-1}(a)} \quad \text{(lr4)}$$

hold for all $a, b \in S$, $u, v \in T$.

The matched product solution $r_S \bowtie r_T$ is given by

$$r_S \bowtie r_T\left((a, u), (b, v)\right) = \left(\left(\alpha_u(b), \beta_a(v) \triangleleft u\right), \left(\alpha^{-1}_{v \triangleleft u}\alpha_u(b) \triangleleft a, \beta_b^{-1}(u)\right)\right),$$

and $r_S \bowtie r_T$ is left non-degenerate if and only if $(T, \triangleleft)$ is right rack.

The matched product of left and right shelves

- $(S, \triangleright)$ is a left shelf
- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}(S)$ a map

- $(T, \triangleleft)$ is a right shelf
- r_T solution associated with $(T, \triangleright)$
- $\beta : S \to \mathrm{Sym}(T)$ a map

$(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions if and only if α_u are homomorphisms of left shelves and β_a are homomorphisms of right shelves

$$\alpha_{u \triangleleft v} = \alpha_u \alpha_v \alpha_u^{-1} \qquad \text{(lr1)} \qquad\qquad \beta_{a \triangleright b} = \beta_b^{-1} \beta_a \beta_b \qquad \text{(lr2)}$$

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$$r_S \bowtie r_T \left((a, u), (b, v)\right) = \left(\left(\alpha_u(b), \beta_a(v) \triangleleft u\right), \left(\alpha_{v \triangleleft u}^{-1} \alpha_u(b) \triangleleft a, \beta_b^{-1}(u)\right)\right),$$

and $r_S \bowtie r_T$ is left non-degenerate if and only if $(T, \triangleleft)$ is right rack.

The matched product of left and right shelves

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- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}\,(S)$ a map

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hold for all $a, b \in S$, $u, v \in T$.

The matched product solution $r_S \bowtie r_T$ is given by

$$r_S \bowtie r_T\left((a, u), (b, v)\right) = \left(\left(\alpha_u(b), \beta_a(v) \triangleleft u\right), \left(\alpha_{v \triangleleft u}^{-1} \alpha_u(b) \triangleleft a, \beta_b^{-1}(u)\right)\right),$$

and $r_S \bowtie r_T$ is left non-degenerate if and only if $(T, \triangleleft)$ is right rack.

The matched product of left shelves

An example

- $(S, \triangleright)$ the left shelf defined by $a \triangleright b := a$, for all $a, b \in S$.
  The solution associated with $(S, \triangleright)$ is defined by $r_S(a, b) = (b, b)$, for all $a, b \in S$.
- $(T, \triangleright) = (S, \triangleright)$.
- θ and η be a bijective maps from S into itself.
- $\alpha, \beta : S \to \mathrm{Sym}(S)$ the constant maps with value θ and η respectively.

Then, $(r_S, r_S, \alpha, \beta)$ is a matched product system of solutions.
The solution $r_S \bowtie r_S$ is the map given by

$$\begin{aligned} r((a,u),(b,v)) &= ((\theta(b), \eta(v)), (\theta^{-1}(\theta(b)), \eta^{-1}(\eta(v)))) \\ &= ((\theta(b), \eta(v)), (b, v)) \end{aligned}$$

for all $a, b, u, v \in S$.

$r_S \bowtie r_S$ is associated with a left shelf if and only if $\lambda_{(a,u)}(b, v) = (b, v)$, for all $(a, u), (b, v) \in S \times T$ and, in other words, if and only if $\theta = \eta = \mathrm{id}_S$.

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The solution $r_S \bowtie r_S$ is the map given by

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- $(T, \triangleright) = (S, \triangleright)$.
- θ and η be a bijective maps from S into itself.
- $\alpha, \beta : S \to \mathrm{Sym}(S)$ the constant maps with value θ and η respectively.

Then, $(r_S, r_S, \alpha, \beta)$ is a matched product system of solutions.
The solution $r_S \bowtie r_S$ is the map given by

$$\begin{aligned} r((a,u),(b,v)) &= ((\theta(b), \eta(v)), (\theta^{-1}(\theta(b)), \eta^{-1}(\eta(v)))) \\ &= ((\theta(b), \eta(v)), (b, v)) \end{aligned}$$

for all $a, b, u, v \in S$.

$r_S \bowtie r_S$ is associated with a left shelf if and only if $\lambda_{(a,u)}(b, v) = (b, v)$, for all $(a, u), (b, v) \in S \times T$ and, in other words, if and only if $\theta = \eta = \mathrm{id}_S$.

The matched product of left shelves

An example

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The matched product of left shelves

When is the matched product solution a shelf?

- $(S, \triangleright)$ is a left shelf
- r_S solution associated with $(S, \triangleright)$
- $\alpha : T \to \mathrm{Sym}\,(S)$ a map

- $(T, \triangleright)$ is a left shelf
- r_T solution associated with $(T, \triangleright)$
- $\beta : S \to \mathrm{Sym}\,(T)$ a map

If $(r_S, r_T, \alpha, \beta)$ is a matched product system of r_S and r_T, then the matched solution $r_S \bowtie r_T$ is associated with a left shelf on the cartesian product $S \times T$ if and only if $\alpha_u = \mathrm{id}_S$, for every $u \in T$, and $\beta_a = \mathrm{id}_T$, for every $a \in S$.

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The matched product of left shelves

The structure shelf

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$$(a, u) \triangleright_r (b, v) = (a \triangleright b, u \triangleright v)$$

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The structure shelf of the matched product

- r_S left non-degenerate solution
- $\alpha : T \to \mathrm{Sym}\,(S)$ a map

- r_T left non-degenerate solution
- $\beta : S \to \mathrm{Sym}\,(T)$ a map

If $(r_S, r_T, \alpha, \beta)$ is a matched product system of solutions, then the matched product solution $r := r_S \bowtie r_T$ is left non-degenerate and the structure shelf is

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Thanks for your attention!