New solutions of the Yang-Baxter equation obtained through solutions of the pentagon equation

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The pentagon equation

The study of the pentagon equation (PE) classically originates from the field of Mathematical Physics and it is widely investigated also in Analysis.
The paper [Dimakis, Müller-Hoissen, 2015] can be useful for a brief introduction to this topic.

Recent developments have been provided in [Catino, Mazzotta, Miccoli, 2019], where this equation is dealt with from an algebraic point of view.

Aim of this talk

Show new applications of the PE to set-theoretical solutions of the well-known Yang-Baxter equation. [Catino, Mazzotta, S., work in progress]

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Show new applications of the PE to set-theoretical solutions of the well-known Yang-Baxter equation. [Catino, Mazzotta, S., work in progress]

Set-theoretical solutions of the PE

Given a set S, a map $s : S \times S \to S \times S$ is a *set-theoretical solution of the PE* on S if

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23}$$

where $s_{12} = s \times \mathrm{id}_S$, $s_{23} = \mathrm{id}_S \times s$, and $s_{13} = (\mathrm{id}_S \times \tau) s_{12} (\mathrm{id}_S \times \tau)$ with τ the twist map, i.e., $\tau(x, y) = (y, x)$, for all $x, y \in S$.
We briefly call s a *solution of the PE*.

In particular, as in [Catino, Mazzotta, Miccoli, 2019] we write

$$s\,(a, b) = (a \cdot b,\ \theta_a\,(b)),$$

for all $a, b \in S$, where θ_a is a map from S into itself, for every $a \in S$.
Note that the structure $(S, \cdot)$ is a semigroup.

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A comparison with the PE:

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Remark: A map s is a solution of the PE if and only if $t := \tau s \tau$ is a reversed solution, that is given by

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Examples

- If S is a semigroup and γ an idempotent endomorphism of S then the map $s : S \times S \to S \times S$ given by

$$s(a, b) = (ab, \gamma(b))$$

is a solution of the PE on S but not of the R-PE.

- *Militaru solutions*: Given f and g idempotent maps from a set S into itself such that $fg = gf$. Then the map s given by

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Set-theoretical solutions of the quantum YBE

According to [Drinfel'd, 1992], given a set S, a map $\mathcal{R} : S \times S \to S \times S$ is said to be *a set-theoretical solution of the quantum Yang-Baxter equation* on S, if

$$\mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12} = \mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23}$$

holds, with the same notation adopted for the PE. For simplicity, we call $\mathcal{R}$ a *solution of the QYBE*.

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A special class of solutions

Proposition (Catino, Mazzotta, S., 2019)

Let s be a solution of the PE on a set S defined by $s(a,b) = (ab, \theta_a(b))$. Then, the map s is a solution of the QYBE if and only if the following conditions

$$abc = a\theta_b(c)\, bc \tag{Y1}$$

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are satisfied, for all $a, b, c \in S$. We call s a solution to the QYBE of pentagonal type, *or briefly* a solution P-QYBE.

Analogously, if t is a reversed solution, $t(a,b) = (\theta_b(a), ba)$, then t is a solution of the QYBE if and only if (Y1), (Y2), and (Y3) are satisfied. We call t a *solution to the QYBE of reversed pentagonal type*, or briefly a *solution R-QYBE*.

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Analogously, if t is a reversed solution, $t(a, b) = (\theta_b(a), ba)$, then t is a solution of the QYBE if and only if (Y1), (Y2), and (Y3) are satisfied. We call t a *solution to the QYBE of reversed pentagonal type*, or briefly a *solution R-QYBE*.

A special class of solutions

Proposition (Catino, Mazzotta, S., 2019)

Let s be a solution of the PE on a set S defined by $s(a, b) = (ab, \theta_a(b))$. Then, the map s is a solution of the QYBE if and only if the following conditions

$$abc = a\theta_b(c)\,bc \tag{Y1}$$

$$\theta_a\theta_b = \theta_b \tag{Y2}$$

$$\theta_a(bc) = \theta_{\theta_b(c)}(bc) \tag{Y3}$$

are satisfied, for all $a, b, c \in S$. We call s a solution to the QYBE of pentagonal type, *or briefly* a solution P-QYBE.

Analogously, if t is a reversed solution, $t(a, b) = (\theta_b(a), ba)$, then t is a solution of the QYBE if and only if (Y1), (Y2), and (Y3) are satisfied. We call t a *solution to the QYBE of reversed pentagonal type*, or briefly a *solution R-QYBE*.

Examples

- *Militaru solutions*: If f and g idempotent maps from a set S into itself such that $fg = gf$, the map

$$s(a,b) = (f(a), g(b))$$

is a solution P-QYBE on S. In this case the semigroup operation is defined by $ab := f(a)$. Clearly, s lies in the class of the well-known *Lyubashenko solutions*.

- If S is such that $abc = adbc$, for all $a, b, c, d \in S$ (cf. [Monzo, 2003]), then

$$s(a,b) = (ab, \gamma(b))$$

with γ an idempotent endomorphism, is a solution to the P-QYBE on S.

- If S is such that $abc = abdbc$ and $a^3 = a^2$, for all $a, b, c, d \in S$, $k \in S$, then the map

$$s(a,b) = (ab, k^2)$$

is a solution of the PE on S but not of the YBE since (Y1) does not hold.

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Solutions of the P-YBE on particular semigroups

We focus on solutions $s(a,b) = (ab, \theta_a(b))$ of the PE defined on specific varieties of semigroups S with the property

$$abc = adbc$$

for all $a, b, c, d \in S$. We are interested in analysing the powers of the solutions of the *"braid version"* of the P-QYBE, i.e.,

$$r(a,b) := \tau s(a,b) = (\theta_a(b), ab).$$

Given a set S, a map $\mathcal{R} : S \times S \to S \times S$ into itself is a solution to the QYBE on S if and only if the map $r := \tau\mathcal{R}$ satisfies the *braid equation*, i.e.,

$$(r \times \mathrm{id}_S)(\mathrm{id}_S \times r)(r \times \mathrm{id}_S) = (\mathrm{id}_S \times r)(r \times \mathrm{id}_S)(\mathrm{id}_S \times r).$$

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The powers of these solutions P-YBE

We show that the solutions P-YBE on these semigroups lie in a special class of solutions of the Yang-Baxter equation.

Theorem (Catino, Mazzotta, S., 2019)

Let S be a semigroup with the property $abc = adbc$ and r a (braid) solution P-YBE on S. Then, it holds

$$r^5 = r^3$$

and the powers r^2, r^3, r^4 of the map r are still solutions to the YBE.

Remark - Example

If S is a left quasi normal semigroup, i.e., $abc = acbc$, then the map on S defined by $r(a, b) := (b, ab)$ is a solution of the P-YBE such that $r^5 = r^3$. If S is not idempotent, then r^2, r^3, r^4 are not solutions of the YBE.

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A new method to construct solutions to the YBE

We introduce a new method to construct solutions of the Yang-Baxter equation defined on the Cartesian product of two sets S and T through solutions of the pentagon equation.

In particular, we show how to obtain a solution of the YBE involving a solution s of the PE and a solution t of the R-YBE.

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A new construction - I

We introduce the following definition.

Definition

Let S, T be semigroups, s a solution of the PE on S and t a solution R-YBE on T. Let $\alpha : T \to S^S$ be a map, set $\alpha_u := \alpha(u)$, for every $u \in T$, and set

$$a_u b_v := \alpha_u(a)\,\alpha_{\theta_v(u)}(b),$$

for all $a, b \in S$ and $u, v \in T$. If the following conditions hold

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for all $a, b, c \in S$ and $u, v, w \in T$, then we call (s, t, α) a *pentagon triple*.

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A new construction - II

Theorem (Catino, Mazzotta, S., 2019)

Let (s, t, α) be a pentagon triple. Then the map given by

$$r(a, u\,;\, b, v) = (\theta_a\alpha_u(b), vu\,;\, a\alpha_u(b), \theta_v(u)),$$

for all $(a, u), (b, v) \in S \times T$ is a solution of the YBE.

This result is a special case of a more general construction.

A new construction - II

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Example

Consider

- S a semigroup with the properties $abdbc = abc$ and $a^3 = a^2$, $k \in S$, and $s(a,b) = (ab, k^2)$ the solution of the PE on S (it is not a solution to the QYBE);
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- $\alpha_u(a) = k^2$, for every $a \in S$ and $u \in T$.

Hence, the map given by

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Thanks for your attention!