Interaction Between Convergence Spaces and Discrete Groups

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Generating set Γ of a group G

Let Γ be a subset of a group G such that each element of G is a product of elements of Γ and no element of Γ is **redundant**.

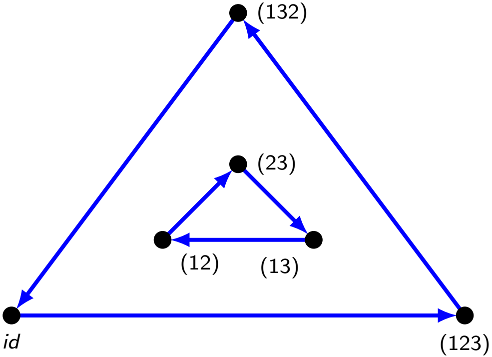
Edge-colored Cayley digraph

The Cayley digraph for G generated by Γ is the directed graph C such that the vertex set of C is G and the edge set of C is $E = \{(g, g\gamma) : g \in G \ , \ \gamma \in \Gamma\}$. The edges are colored by $j : E \to \Gamma$, where $j(g, h) = s$.

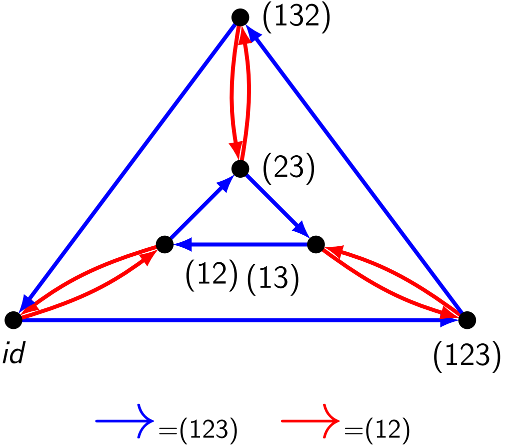
Consider the symmetric group S_3 with generators $\{(12), (123)\}$.

Generator	id	(12)	(13)	(23)	(123)	(132)
(12)	(12)	id	(123)	(132)	(13)	(23)
(123)	(123)	(23)	(12)	(13)	(132)	id

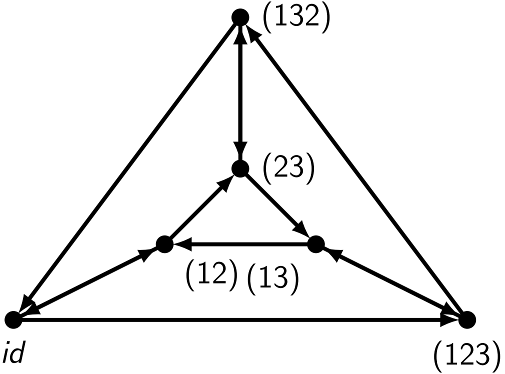
Generator	id	(12)	(13)	(23)	(123)	(132)
(123)	(123)	(23)	(12)	(13)	(132)	id



Edge-colored Cayley digraph of S_3

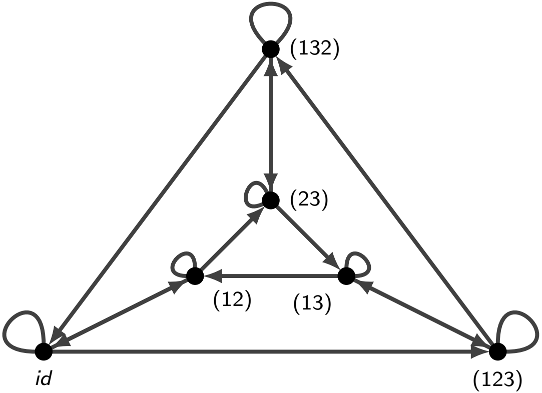


Cayley digraph (without color)



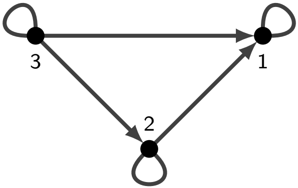
Reflexive Cayley digraph[1]

The Cayley graph for G generated by Γ is the reflexive digraph C such that the vertex set of C is G and the edge set of C is $\{(g, h) : g\gamma = h \wedge (\gamma = e \vee \gamma \in \Gamma)\}$.



[1] [Definition 3.20], D.R.Patten, Problems in the theory of convergence spaces. 2014, Thesis.

Convergence generated by a reflexive digraph



The graph neighbourhood[2] of the vertices are
$\overrightarrow{1} = \{1\}, \quad \overrightarrow{2} = \{1,2\}, \quad \overrightarrow{3} = \{1,2,3\}$.
This graph can be represented as the following convergence:

$$\begin{array}{lll} \{1\}^{\uparrow} \to \{1,2,3\} & \{1,2\}^{\uparrow} \to \{2,3\} & \{1,2,3\}^{\uparrow} \to \{3\} \\ \{2\}^{\uparrow} \to \{2,3\} & \{1,3\}^{\uparrow} \to \{3\} & \\ \{3\}^{\uparrow} \to \{3\} & \{2,3\}^{\uparrow} \to \{3\} & \end{array}$$

No topology can describe this convergence.

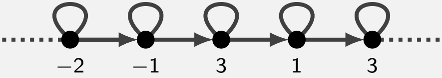
[2] [Definition 3.1], D.R.Patten (2014), Problems in the theory of convergence spaces. Thesis.

Convergence space[3]

Let λ be an arbitrary relation between X and the power set of, set of all filters on X. The relation is called convergence on that set if for $\mathcal{F}_1, \mathcal{F}_2$ in $\mathbb{F}X$ and x in X the following conditions hold:

(i) **Centred:** $x^{\uparrow} \in \lambda(x)$,
(ii) **Isotone**: If $\mathcal{F}_1 \in \lambda(x)$ and $\mathcal{F}_1 \leq \mathcal{F}_2$ then $\mathcal{F}_2 \in \lambda(x)$, and
(iii) **Finitely deep**: If $\mathcal{F}_1, \mathcal{F}_2 \in \lambda(x)$ then, $\mathcal{F}_1 \cap \mathcal{F}_2 \in \lambda(x)$.

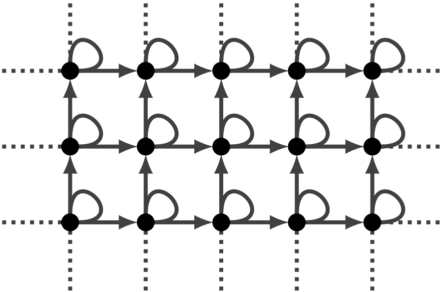
Category of convergence spaces contain category of reflexive directed graphs.



Reflexive Cayley digraph of $\mathbb{Z}$ generated by $\{1\}$

[3] S. Dolecki and F. Mynard, Convergence Foundations of Topology, World Scientific Publishing Company, 2016.

A Cayley graph for $\mathbb{Z} \oplus \mathbb{Z}$ is the graph Cartesian product $\mathbb{Z} \times \mathbb{Z}$ generated by $(\Gamma \times \{e\}) \cup (\{e\} \times \Gamma)$.[4]



Define a function $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ as $+(a, b) = a + b$.
Clearly, $+$ is continuous.

[4] [Example 4.6], Patten et. al, Differential calculus on Cayley graphs, 2015

Convergence Groups

A convergence group is a group endowed with a convergence structure such that group operations are continuous in the sense of convergence.

Remark

- The equivalence between the Cayley graphs and the convergence spaces can be used to construct the convergence groups beyond the class of homeomorphism groups.
- Convergence spaces are used in unifying discrete and continuous models of computation and they play a vital role in extending the definition of differential to the discrete structures.
- Convergence spaces play a prominent role in extending the Pontryagin duality theorem beyond local compactness.

Dual group

Circle group , $\mathbb{T} \cong \mathbb{R}/\mathbb{Z} \cong U(1)$

The multiplicative group of all complex numbers of unit modulus with the natural topology as a subspace of the complex plane.

Character group, $\hat{G}$ or $\mathbb{C}Hom(G, \mathbb{T})$

The homomorphism, $\chi : G \to \mathbb{T}$ is called a character and the set of all continuous characters, of an **abelian** group with the operation of pointwise multiplication is called character group.

Dual group, $(\hat{G}, \tau_{co})$

The character group with compact open topology is called the dual group.

Pontryagin-van Kampen theorem

Pontryagin duality

For a topological abelian group there is a natural evaluation homomorphism from the group to its double dual defined by

$$\alpha_G : G \to \hat{\hat{G}} \qquad \alpha_G(g)(\chi) = \chi(g) \quad \forall\, g \in G.$$

If this evaluation map is a topological isomorphism then the group is said to satisfy Pontryagin duality or is said to be **Pontryagin reflexive**.

Pontryagin-van Kampen theorem[5]

Every locally compact abelian (**LCA**) group is canonically isomorphic to its double dual group.

[5] van-Kampen, R. E. (1935). Locally bicompact abelian groups and their character groups. Ann Math, 97:448-463.

Moving Beyond Topology?

A topology on $\mathbb{C}Hom(G, \mathbb{T})$ is called `admissible` if the evaluation mapping

$$e : \mathbb{C}Hom(G, \mathbb{T}) \times G \to \mathbb{T}, \quad e(\chi, g) = \chi(x)$$

is continuous.

Reflexive Admissible Topological Group[6]

If G is a reflexive topological abelian group, then the evaluation mapping is continuous if and only if G is locally compact.

Topological structures are inadequate for situations in analysis particularly when we go beyond local compactness.

Beyond topology - Convergence spaces

[6] Martin-Peinador,E.(1995). A reflexive admissible topological group must be locally compact. Proc Amer Math Soc, 123(11):3563-3566.

Continuous convergence structure λ_c

The continuous convergence structure on the character group of a convergence abelian group is the coarsest convergence structure which makes the evaluation mapping $e : \mathbb{C}Hom(G, \mathbb{T}) \times G \to \mathbb{T}$ continuous.

Continuous dual group, $\Gamma_c G$

The character group with continuous convergence structure is called the dual group.

Continuous duality [7]

A convergence group is c-reflexive if the mapping

$$\kappa : G \to \Gamma_c \Gamma_c G$$

defined by

$$\kappa(g)(\chi) = \chi(g) \; \forall \, g \in G, \; \chi \in \Gamma_c G$$

is a continuous group homomorphism, here $\Gamma_c G = (\mathbb{C}Hom(G, \mathbb{T}), \lambda_c)$

[7] H.-P. Butzmann, Duality theory for convergence groups, Topology Appl. 111 (2000), no. 1, 95–104.

Example

Non c-reflexive locally compact convergence group[8]

Let X be a locally compact topological space, $C(X)$ and $C(X, \mathbb{T})$ respectively denote the group of all continuous, real-valued functions on X and the group of unimodular $(X \to \mathbb{T})$ continuous functions on X.
Define

$$\rho : C_c(X) \to C_c(X, \mathbb{T}) \text{ as } \rho(f) = \rho \circ f.$$

- ρ is continuous and a group homomorphism.

As X is locally compact so

$$C_c(X) = C_{co}(X) \text{ and } C_c(X, \mathbb{T}) = C_{co}(X, \mathbb{T}).$$

- $C_c(X, \mathbb{T})$ is reflexive.

[8] Beattie,R. and Butzmann,H.-P.(2013). Convergence Structures and Applications to Functional Analysis. Bücher. Springer Netherlands, 2013.

For

$$\kappa : X \to \Gamma_c C(X, \mathbb{T}) \text{ defined as } \kappa(x)f = f(x)$$

we have,

Theorem[9]

The group generated by $\kappa(X)$, (denoted $G =< \kappa(X) >$) is a locally compact subgroup of $\Gamma_c C_c(X, \mathbb{T})$.

Example[10]

For X a connected, compact topological space, the group $G =< \kappa(X) >$ is not reflexive.

Problem

- To characterise the class of **reflexive locally compact convergence groups**.

[9] Proposition 8.5.12, Beattie,R. and Butzmann,H.-P.(2013). Convergence Structures and Applications to Functional Analysis. Bücher. Springer Netherlands, 2013.

[10] Example 8.5.14, Beattie,R. and Butzmann,H.-P.(2013). Convergence Structures and Applications to Functional Analysis. Bücher. Springer Netherlands, 2013.

Approach to solve the problem

- Local quasi convexity.
  - It has been obtained that local quasi convexity is a necessary condition for a locally compact convergence group to be c-reflexive.
  - Non-topological compact convergence groups **(if they exist)** are reflexive iff they are locally quasi convex.
- Convergence measure spaces.
  - The problem is to obtain the notation of integration on convergence spaces and hence a notation similar to Haar measure for the convergence groups.
  - Are the convergence groups whose topological modification locally compact topological group reflexive?
- Bounded convergence groups.