On the Normal Length of a Group

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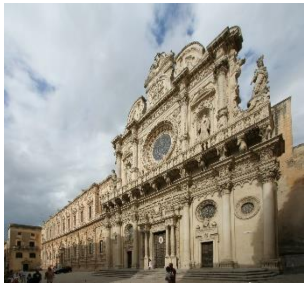

Motivations

Let G be a group and let X be a subgroup of G. Denote by $[X^G/X]$ the interval of the subgroup lattice consisting of all subgroups Y such that $X \leq Y \leq X^G$.

What can we say about the structure of G if the interval $[X^G/X]$ is "small", for every subgroup X of G?

(R. Dedekind, 1897 - R. Baer, 1933) *Let G be a group. Then G is a Dedekind group if and only if G is either abelian or can be decomposed as the direct product of the quaternion group Q_8 of order 8 and a periodic abelian group with no elements of order 4.*

A group G is said to be an FC-group if each of its elements has only finitely many conjugates. Recall that an FC-group G is locally finite over its centre. In particular, its commutator subgroup G' is locally finite.

A group G is an FC-group if and only every cyclic subgroup of G has finite index in its normal closure.

(B.H. **Neumann, 1955)** *Let G be a group in which every subgroup has finite index in its normal closure. Then the commutator subgroup G′ of G is finite.*

In particular,

$$|X^G : X| \leq |G'|,$$

for each subgroup X of G.

➢ F. **de Giovanni** - A. **R.**: "Groups of finite normal length", *Bull. Aust. Math. Soc.* 97 (2018), 229-239.

Definition - *A subgroup X of a group G is said to have normal length k in G if there exists a non-negative integer k such that all chains between X and its normal closure X^G have length at most k and k is the length of at least one of these chains.*

The group G is said to have a finite normal length if there is a finite upper bound for the normal length of its subgroups. The least upper bound is called the normal length of G.

Notation

nl(X,G) (normal length of X in G)

nl(G) (normal length of G)

Examples and Remarks

Let G be any group and let X be a subgroup of G.

- *$nl(X, G)=0$ iff X is normal in G.*
- *$nl(X,G)=1$ iff X is a maximal subgroup of X^G .*
- *$nl(G)=0$ iff G is a Dedekind group.*
- *If G is a Tarski p-group, then $nl(G)=1$.*
- *If G' has order prime, then $nl(G)\leq 1$.*
- *If each subgroup of G is nearly normal, then G has finite normal length.*

- The structure of groups in which every *cyclic* subgroup has normal length at most 1 (*J-groups*) was investigated by M. **Herzog,** P. **Longobardi,** M. **Maj** and A. **Mann** in 2000.

Among the results obtained:

✓ *A locally soluble (locally finite, respectively) J-group is soluble with derived length at most 3.*

✓ *An infinite simple group G is a J-group if and only if all proper subgroups of G are abelian.*

Locally soluble groups

It can be proved that the cyclic subgroups of finite normal length in a locally soluble group are nearly normal.

1. (F. **de Giovanni** -A. **R., 2018)** *Let G be a locally (soluble-by-finite) group in which all subgroups have finite normal length. Then the commutator subgroup G′ of G is finite, and so G has finite normal length.*

Let X be an ascendant subgroup of a group G such that $nl(X,G)=k$. It is easy to prove that X is subnormal with defect at most $k+1$.

(J. E. **Roseblade**, 1965) *A group all of whose subgroups are subnormal with defect at most k (for some fixed positive integer k) is nilpotent with nilpotency class at most* $\xi(n)$*, for a suitable function* ξ*.*

Clearly,

$$\xi(1)=2,$$

while,

$$\xi(2)=3$$

(H. **Heineken**, 1971; S. K. **Mahdavianary**, 1983).

2. (F. **de Giovanni -** A. **R., 2018)** *Let G be a locally nilpotent group of finite normal length k. Then G is nilpotent and its nilpotency class is at most* $\xi(k+1)$.

Let G be a group with normal length at most 2. Then every subnormal subgroup of G has defect at most 3.

(T. **Hawkes**, 1984) *Every finite soluble group can be embedded in a finite soluble group whose subnormal subgroups have defect at most 3.*

Therefore the derived length of such groups cannot be bounded.

3. (F. de Giovanni - A. R., 2018) *Let G be a locally soluble group of finite normal length k. Then G is soluble and its derived length is at most $\psi(k)$, for suitable function ψ.*

(H. **Zassenhaus**, 1938) *If G is a soluble linear group of degree n over an arbitrary field, then there exists a function θ such that the derived length of G is bounded by $\theta(n)$.*

Let N be a minimal normal subgroup of a finite soluble group of finite length k, and let x be an element in N. Clearly, N is abelian of prime exponent p and $\langle x \rangle^G = N$. Therefore $|N| \le p^{k+1}$ and hence the derived length of $G/C_G(N)$ is at most $\theta(k+1)$.

The function ψ in Theorem 3 is defined by the position

$$\psi(k)=\theta(k+1)+[\log_2(\xi(k+1))]$$

where θ and ξ are the functions of Zassenhaus and Roseblade, respectively.

Simple groups

A finite group of normal length 2 cannot be simple.

4. (F. de Giovanni – A. R., 2018) *Let G be a finite simple non-abelian group. Then* $nl(G) \geq 3$. *As a consequence, a finite group with normal length at most 2 is soluble.*

Note that the alternating group *Alt(5)* has normal length *3* and

$$nl(Alt(n)) < nl(Alt(n+1))$$

for all $n \geq 5$.

Therefore the normal length of finite simple group cannot be bounded.

Definition – *Let n be a positive integer. An infinite simple group G is said to be a Tarski n-monster if it satisfies both the minimal and the maximal conditions on subgroups, every proper subgroup of G can be generated by at most n elements, and n is the smallest positive integer with such property.*

Note that the Tarski *1*-monsters are precisely the ordinary Tarski groups (**Ol'shanskii**, 1979). Any Tarski n-monster is finitely generated and has finite rank, either n or $n+1$.

5 (F. de Giovanni - A. R., 2018) *Let G be an infinite simple group of finite normal length k. Then G is a Tarski n-monster for some positive integer $n \leq k$.*

Examples

(V.N. Obraztsov, 1989) *Let* $\{G_i \mid i \in I\}$ *be a set of non-trivial finite or countable groups* G_i *without involutions such that* $2 \leq |I| \leq \omega$*. Suppose that n is a sufficiently large odd number (for example,* $n > 2 \times 10^{77}$ *). Then there exists a countable simple group G containing a copy of* G_i *for all i, with the following properties:*

- $G_i \cap G_j = \{1\}$, *whenever* $i \neq j$;

- *If* x *and* y *are element of* G *such that* $x \in G_i$ *and* $y \in G \setminus G_i$ *for some* i, *then* $G = \langle x, y \rangle$;

- *Every proper subgroup of* G *is either cyclic of order dividing* n *or is contained in a conjugate of some* G_i.

Example 1 *For each prime number* $p > 2\times 10^{77}$ *there exists a simple two-generator infinite p-group G such that its normal length is infnite, but every subgroup has finite normal length in G.*

Proof – It is enough to apply the quoted result of Obraztsov to the set $\{G_k \mid k \in N\}$, where every G_k is a finite p-group containing a (subnormal) subgroup with defect at least $k+1$. □

Example 2 *For each prime number $p > 2 \times 10^{77}$ and each positive integer k there exists a simple two-generator infinite p-group G_k of normal length k and G_k is a Tarski k-monster.*

Proof - Denote by G_1 a Tarski p-group, and suppose that a simple p-group G_k of normal length k has been constructed for some $k \geq 1$. Let A_k be an abelian group of exponent p and order p^{k+1}.

If we apply the quoted result of Obraztsov to the triple (G_k, A_k, p) we obtain a simple two – generator group G_{k+1} in which every proper non-cyclic subgroups is contained either in a conjugate of G_k or in conjugate of A_k. In particular, A_k and G_k are maximal subgroups of G_{k+1}. Therefore G_{k+1} has normal length $k+1$, and it is a $(k+1)$-monster since the subgroup A_k is generated by $k+1$ elements. □

Some related Problems

- *Groups in which every abelian subgroup have finite normal length.*

(recall that a group G is an FC-group if and only if every abelian subgroup is nearly normal (M. J. **Tomkinson**, 1981) .

- *Groups in which every non-abelian subgroup have finite normal length.*
- ……

Many Thanks

Further Results

Recall that a group is said to be *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index.

6. **(**F. **de Giovanni -** A. **R., 2018)** *Let G be a locally graded group of normal length at most 2. Then G is soluble.*

A group G is said to have a *Sylow tower* if it admits an ascending normal series

$$\{1\}=G_0<G_1<\cdots<G_\alpha<\cdots<G_\mu<G_{\mu+1}=G$$

such that $G_{\alpha+1}/G_\alpha$ is a Sylow subgroup of G/G_α for each ordinal $\alpha<\mu$ and G/G_μ is torsion-free. In particular, a periodic group G has a Sylow tower iff every non-trivial homomorphic of G contains a non-trivial normal Sylow subgroup. Clearly, every finite group with a Sylow tower is soluble and p-nilpotent for some prime p. A classical result of G. **Zappa** states that a finite supersoluble group has a Sylow tower.

7. (F. **de Giovanni -** A. **R., 2018)** *Let G be a locally graded group of normal length at most 2. Then G has a Sylow tower.*

Note that the group ***Alt(4)×Sym(3)*** has normal length 3, but it does not have a Sylow tower.