L-algebras and their groups

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How is it possible that a mathematical structure with a single binary operation, based on a single equation (associativity) appears on every showplace in mathematics, most often in an essential way?

To be sure: We are talking about **groups**! — Are there other structures of that kind?

1. L-algebras and logic

Given that groups are invincible, let us exhibit a structure with a single operation, based, too, on a single equation, less trivial than associativity, a structure that contributes a missing aspect to many groups: **order**. Just as associativity allows to built finite strings, the **cycloid equation**

$$\boxed{(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)} \qquad \text{(L)}$$

gives a blueprint for infinite braid-like structures. It occurs in several ways in connection with **right ℓ-groups** (e. g., Garside groups and various function spaces), geometry, and quantum theory.

The “L” stands for **logic**: Replacing $\cdot$ by an arrow for “implication”, (L) asserts the equivalence (“=”) of two logical propositions:

$$\boxed{(x \to y) \to (x \to z) = (y \to x) \to (y \to z)} \quad \text{(L)}$$

To make the **operation** “$\to$” into a relation “$\leqslant$” (x **entails** y), we need an element 1 which stands for **truth**: x entails y if and only if $x \to y$ is true:

$$x \leqslant y \;:\Longleftrightarrow\; x \to y = 1.$$

A **logical unit** 1 has to satisfy

$$\boxed{1 \to x = x, \qquad x \to x = x \to 1 = 1} \qquad \text{(U)}$$

From (L) and (U) it follows that entailment $\leqslant$ is reflexive and transitive. To get a partial order, we assume

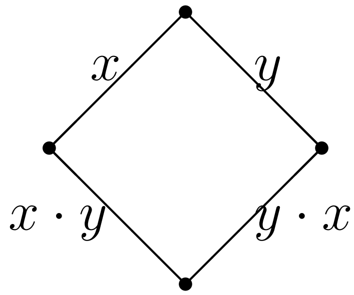
$$\boxed{x \to y = y \to x = 1 \implies x = y} \qquad \text{(E)}$$

Definition 1. A set $(X; \to)$ with (L), (U), and (E) is said to be an ***L*-algebra**.

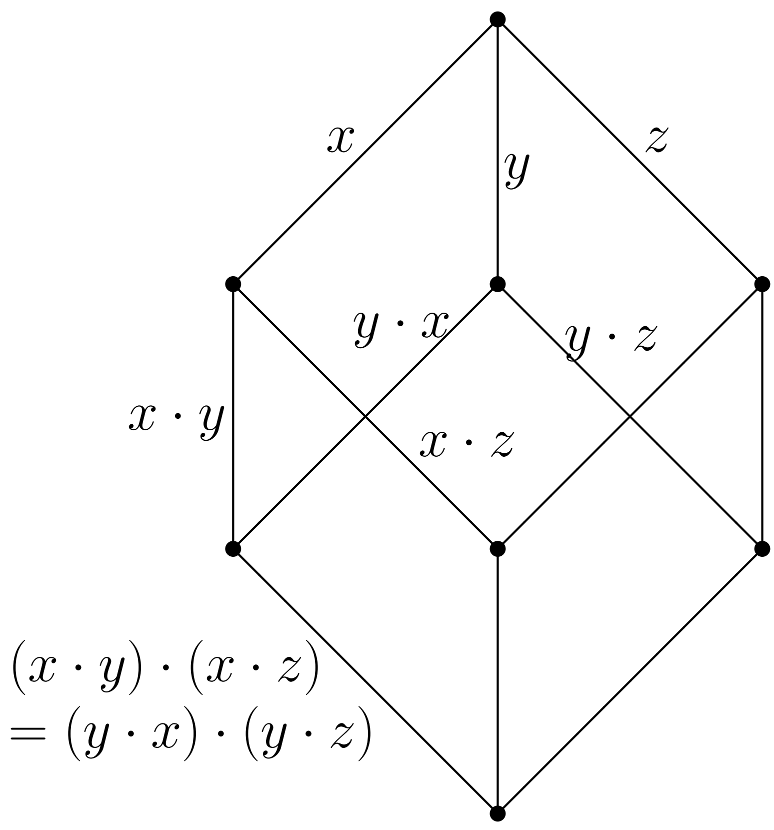
Thus every L-algebra comes with a partial order. The element 1 is always the greatest element of X.

Definition 2. An L-algebra X is **discrete** if the elements in $X \smallsetminus \{1\}$ are pairwise incomparable.

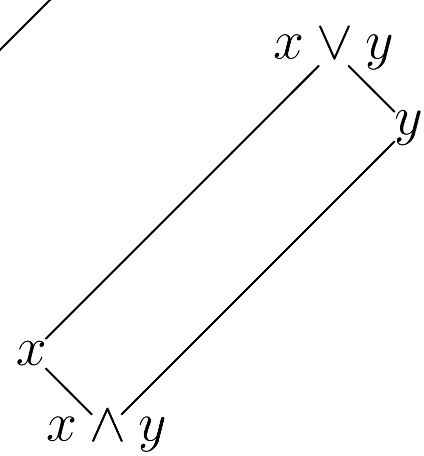
Let $(X;\cdot)$ be a **discrete** L-algebra. For any pair of distinct $x, y \in S^1(X) := X \smallsetminus \{1\}$ we built a mesh



and **iterate** the procedure. Eq. (L) guarantees that the construction yields a lower semimodular lattice. The generic case looks as follows:



Recall that a lattice is said to be **lower semimodular** if whenever $x \vee y$ covers y, then x covers $x \wedge y$.



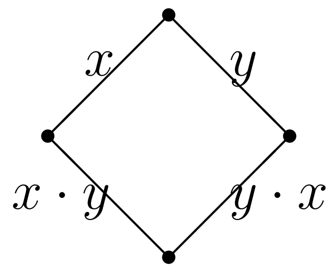
We obtain a labelled lattice, an L-algebra which can be regarded as the Cayley graph of a monoid $S(X)$, the **self-similar closure** of X. Now this construction generalizes to arbitrary L-algebras.

Definition 3. An L-algebra $(X;\rightarrow)$ is said to be **self-similar** if for all $x, y \in X$ there is an element $z \leqslant y$ with $y \rightarrow z = x$.

Such an element z depends uniquely on x and y. The new operation $xy := z$ is then **associative**! Moreover,

$$\boxed{xy \rightarrow z = x \rightarrow (y \rightarrow z)} \tag{A}$$

Thus, logically, the multiplication stands for a non-commutative **conjunction**. The mesh relation



leads to another, commutative operation

$$\boxed{x \wedge y := (x \rightarrow y)x = (y \rightarrow x)y} \tag{H}$$

which makes X into a $\wedge$-semilattice. Thus $x \wedge y$ gives the **classical conjunction**. In what follows, we return to our former notation, writing $\cdot$ instead of $\rightarrow$. Replacing $xy \cdot z$ in (A) by $x \cdot yz$, we have the following cocycle equation

$$x \cdot yz = \big((z \cdot x) \cdot y\big)(x \cdot z), \tag{S}$$

which is equivalent to the first of the equations

$$x{\cdot}yx = y \tag{I}$$

$$xy \cdot z = x \cdot (y \cdot z) \tag{A}$$

$$(x \cdot y)x = (y \cdot x)y \tag{H}$$

Proposition 1. *A self-similar L-algebra X is equivalent to a monoid with a second operation · satisfying (I), (A), and (H).*

The unit element of the monoid is the logical unit 1. Note that (A) and (H) imply (L):

$$(x{\cdot}y){\cdot}(x{\cdot}z) \overset{(A)}{=} (x{\cdot}y)x{\cdot}z \overset{(H)}{=} (y{\cdot}x)y{\cdot}z \overset{(A)}{=} (y{\cdot}x){\cdot}(y{\cdot}z).$$

The implication

$$\boxed{x \cdot y = y \cdot x = 1 \implies x = y} \tag{E}$$

can be obtained from the equations as follows:

$$x = 1x = (x \cdot y)x \overset{(H)}{=} (y \cdot x)y = 1y = y.$$

Theorem 1 (2008). *Every L-algebra X is an L-subalgebra of a self-similar L-algebra S(X), so that X generates the monoid S(X). These two properties determine S(X), up to isomorphism.*

$S(X)$ is called the **self-similar closure** of X. Thus any L-algebra embeds into a bigger structure $S(X)$ with more operations to simplify calculations. For example, the $\wedge$-operation satisfies

$$x \cdot (y \wedge z) = (x \cdot y) \wedge (x \cdot z) \tag{1}$$
$$(x \wedge y) \cdot z = (x \cdot y) \cdot (x \cdot z), \tag{2}$$

a commutative version of

$$x \cdot yz = \big((z \cdot x) \cdot y\big)(x \cdot z) \tag{S}$$
$$xy \cdot z = x \cdot (y \cdot z). \tag{A}$$

The equation

$$\boxed{(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)} \tag{L}$$

has the remarkable property that it extends from any set X to the free monoid $M(X)$, using only the equations (S) and (A), and $1 \cdot x = x$. For an L-algebra X, this can be used to construct the self-similar closure by a surjection $M(X) \twoheadrightarrow S(X)$.

Definition 4. An L-algebra X is **$\wedge$-closed** if it is closed with respect to $\wedge$ in $S(X)$.

The $\wedge$-closure $C(X)$ in $S(X)$ is again an L-algebra. Moreover, there is a simple characterization:

Proposition 2. *An L-algebra X is $\wedge$-closed if and only if it satisfies Eqs.* (1) *and* (2)*.*

2. The structure group

An L-algebra X is self-similar iff $S(X) = X$. Then

$$x \cdot yx = y \tag{I}$$

implies that X is right cancellative. By

$$(x \cdot y)x = (y \cdot x)y, \tag{H}$$

X satisfies the left Ore condition. So X has a group $G(X)$ of left fractions $x^{-1}y$ $(x, y \in X)$. For an arbitrary L-algebra X, we call $G(X) := G(S(X))$ the **structure group** of X.

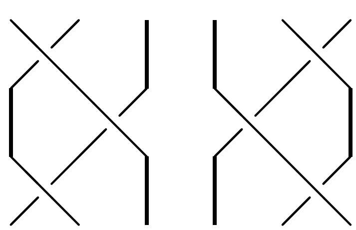
Question: Which groups arise as the structure group of an L-algebra?

Theorem 2 (2016). *The structure group of an L-algebra is torsion-free.*

Example 1. The **braid group** B_n with n strings is a structure group. For example, consider the two generators of B_3:

x $\qquad\qquad$ y

Then



$xyx = yxy$

The braid group B_3 is the structure group of the L-algebra $X = \{1, x, y, xy, yx\}$, given by

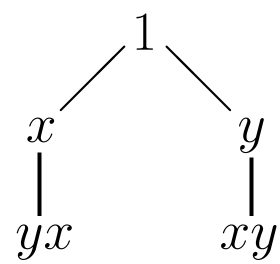
$$x \cdot y := xy, \qquad y \cdot x := yx.$$

For example,

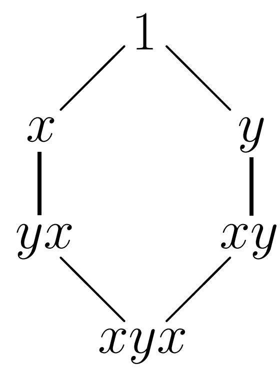
$$x \cdot xy \overset{(S)}{=} \big((y \cdot x) \cdot x\big)(x \cdot y) = (yx \cdot x)xy = 1xy = xy$$

$$yx \cdot xy = y \cdot (x \cdot xy) = y \cdot xy \overset{(I)}{=} x.$$

The partial order of X is given by



The **∧-closure** $C(X)$ is a lattice ("benzene ring"):



an ***L*-algebra with zero**. (A smallest element in a lattice is usually denoted by 0.)

Similarly, every **finite Coxeter group** gives rise to an L-algebra with 0, and with the corresponding Artin group as its structure group.

The braid group B_n is a **right ℓ-group**, that is, a group with a lattice order satisfying

$$(x \vee y)z = xz \vee yz.$$

If $z(x \vee y) = zx \vee zy$ also holds, the group is said to be **lattice-ordered** or briefly, an **ℓ-group**.

Example 2. The **negative cone**

$$G^- := \{x \in G \mid x \leqslant 1\}$$

of a right ℓ-group G is a self-similar L-algebra:

$$x \cdot y := yx^{-1} \wedge 1.$$

Therefore, any right ℓ-group is a structure group (of its negative cone). Indeed, we even have

Proposition 3. *Any right ℓ-group is a two-sided group of fractions of its negative cone.*

In particular, any right ℓ-group is a structure group, hence torsion-free. For a while, it was not known whether the braid group B_n is torsion-free. This was first proved by Fadell, Fox, and Neuwirth (1962) by topological arguments. Direct proofs were given by Rolfsen-Zhu (1998) and Dehornoy (1998, 2004), using the Garside structure.

With the concept of right ℓ-group, a one-line proof becomes possible:

Proposition 4. *Any right ℓ-group is torsion-free.*

Proof. If $g^n = 1$, then $h := 1 \vee g \vee \cdots \vee g^{n-1}$ satisfies $hg = h$. Whence $g = 1$. □

Note that braid groups are right ℓ-groups, but the structure group of an L-algebra need not even carry a partial order.

3. Commutative L-algebras

An element g of a right ℓ-group G is said to be **normal** if it satisfies $g(x \wedge y) = gx \wedge gy$ for all $x, y \in G$. The normal elements form an ℓ-group $N(G)$, the **quasi-centre** of G. For a braid group B_n, the quasi-centre is $\langle 0 \rangle$, the infinite cyclic group generated by the smallest element 0 of its L-algebra. The centre of B_n is $\langle 0^2 \rangle$.

Definition 5. A normal element u of a right ℓ-group G is said to be a **strong order unit** if every $x \in G$ is majorized by some u^n with $n \in \mathbb{N}$.

Examples. In the abelian ℓ-group of continuous functions on a compact space, the positive constants are strong order units. In a braid group B_n, the **Garside element** 0^{-1} is a strong order unit.

Since each L-algebra embeds into a monoid, we have a natural commutativity concept for L-algebras:

Definition 6. Let X be an L-algebra. We say that X is **commutative** if its self-similar closure $S(X)$ is commutative as a monoid.

Commutative L-algebras with 0 are equivalent to **MV-algebras**, introduced by Chang in 1958 as models for **many-valued logic**. (Truth values are in the interval $[0, 1]$ instead of $\{0, 1\}$.)

Viewed as L-algebras, known facts on MV-algebras become more transparent, and new aspects arise.

Proposition 5. *An L-algebra X is commutative if and only if the following are satisfied:*

$$x \leqslant y \cdot x \tag{K}$$

$$x \vee y := (x \cdot y) \cdot y = (y \cdot x) \cdot x \tag{V}$$

Eq. (V) then makes X into a $\vee$-semilattice. If there is a smallest element, X is even a lattice:

Proposition 6. *Let X be an MV-algebra.*

(a) *X is a distributive lattice.*

(b) *$y \mapsto x \cdot y$ is a lattice homomorphism $X \to X$.*

(c) *$x \mapsto x \cdot y$ is a lattice homomorphism $X^{\mathrm{op}} \to X$.*

Mundici proved (1986) that every MV-algebra can be represented as an interval in an abelian ℓ-group.

In terms of L-algebras, this famous result reduces to a property of the structure group:

Theorem 3. *For an MV-algebra X, the natural map $X \to G(X)$ embeds X as an interval $[0,1]$ into $G(X)$, and 0 is a strong order unit in $G(X)$.*

Proof. As a commutative self-similar L-algebra, $S(X)$ is cancellative. Hence $S(X) \to G(X)$ is an embedding. If $x \in X$ and $x \leqslant a \leqslant 1$ in $S(X)$, then $a = a \vee x = (a \cdot x) \cdot x$. By

$$xy \cdot z = x \cdot (y \cdot z) \tag{A}$$

and induction, $a \cdot x \in X$. Whence $a = (a \cdot x) \cdot x \in X$. So X is an interval in $S(X)$, hence in $G(X)$. □

Theorem 3 extends to ℓ-groups $G(X)$ (which gives Dvurečenskij's 2002 generalization) and even to right ℓ-groups (which applies, e. g., to Garside groups and para-unitary groups).

Every MV-algebra X has a natural involution

$$x^* := x \cdot 0$$

which is an **lattice anti-automorphism**:

$$(x \vee y)^* = x^* \wedge y^*$$
$$x^* \cdot y^* = y \cdot x.$$

4. Measure theory

The functorial property of the structure group of an L-algebra is closely related to (commutative or non-commutative) measure theory. In classical terms, a **measure** is a σ-additive function

$$\mu\colon \mathscr{S}(X) \to \mathbb{R}^+$$

from a σ-algebra $\mathscr{S}(X)$ of measurable sets to the non-negative reals. Let us replace the σ-algebra $\mathscr{S}(X)$ by any Boolean algebra. Sometimes it is also more reasonable to work with additive instead of σ-additive measures, or to consider values in the extended reals or in the unit interval $I := [0,1]$. So one would consider a measure

$$\mu\colon \mathscr{B} \to I$$

from a Boolean algebra $\mathscr{B}$ to the unit interval I. Note that both $\mathscr{B}$ and I are MV-algebras. Indeed, a Boolean algebra is equivalent to an MV-algebra satisfying the **sharpness** equation

$$x \cdot (x \cdot y) = x \cdot y$$

This equation implies that $x \cdot x^* = x^*$, which yields $x \vee x^* = (x \cdot x^*) \cdot x^* = x^* \cdot x^* = 1$ and $x \wedge x^* = (x \cdot x^*)x = x^*x = (x \cdot 0)x = 0$. The L-algebra structure of $I = [0,1]$ is given by

$$x \cdot y := \min\{1 - x + y, 1\}.$$

The structure group $G(I)$ of I is the additive group of reals $\mathbb{R}$, with the embedding $I \hookrightarrow \mathbb{R}$ given by $x \mapsto x - 1$. For a Boolean algebra $\mathscr{B}$, the structure group $G(\mathscr{B})$ is a **Specker group**, which can be identified with a group of $\mathbb{Z}$-valued step functions on $\operatorname{Spec}\mathscr{B}$.

Definition 7. Let X, Y be MV-algebras, viewed as subalgebras of $S(X)$ and $S(Y)$. We define a **measure** $\mu\colon X \to Y$ to be a map which satisfies $\mu(xy) = \mu(x)\mu(y)$ for all $x, y \in X$ with $xy \in X$.

The condition $xy \in X$ is equivalent to $y^* \leqslant x$. For a Boolean algebra, this stands for **disjointness** of x and y. An intrinsic condition for measures:

Proposition 7. *A measure $\mu\colon X \to Y$ between MV-algebras is equivalent to a function which satisfies $\mu(x \cdot y) = \mu(x) \cdot \mu(y)$ and $\mu(x) \geqslant \mu(y)$ for all $x \geqslant y$ in X.*

In terms of the structure group:

Theorem 4. *Every measure $\mu\colon X \to Y$ between MV-algebras extends uniquely to a group homomorphism $G(\mu)\colon G(X) \to G(Y)$. Conversely, any group homomorphism $f\colon G(X) \to G(Y)$ with $f(X) \subset Y$ restricts to a measure $\mu\colon X \to Y$.*

The next result interpretes any MV-algebra as a **generalized measure space**. Recall first that every MV-algebra is a distributive lattice. There is a duality

$$\mathrm{Spec}\colon \mathbf{D}^{\mathrm{op}} \longrightarrow \mathbf{Sp} \tag{3}$$

between distributive lattices and **spectral spaces**, the same spaces which also arise as prime spectra of commutative rings.

The functor Spec extends the well-known **Stone duality** between Boolean algebras and Stone spaces. If a spectral space X is endowed with the **patch topology**, we obtain a Stone space $\widetilde{X}$ together with a bijective continuous map $\widetilde{X} \to X$.

For a distributive lattice D, this yields a natural embedding into a Boolean algebra $B(D)$.

Theorem 5. *Let X be an MV-algebra. There is a unique measure $\mu\colon B(X) \twoheadrightarrow X$ with $\mu|_X = 1_X$.*

We call μ the **canonical measure** μ_X of X.

Example 3. The canonical measure of $I := [0, 1]$ is an additive measure $\mu_I\colon B(I) \twoheadrightarrow I$ which uniquely extends to the **Lebesgue measure** on the Borel sets of I.

More group theory is in the wake of MV-algebras.

With measures $\mu\colon X \to Y$ as morphisms, MV-algebras form a category **MV**. For any μ, we call

$$\operatorname{Ker}\mu := \{x \in X \mid \mu(x) = 1\}$$

the **kernel** of μ. To understand the next result, we mention that there is a concept of **ideal** for any L-algebra X, so that ideals I of X correspond to surjective morphisms $X \twoheadrightarrow X/I$.

Proposition 8. *Let $\mu\colon X \to Y$ be a measure of MV-algebras. Then $\operatorname{Ker}\mu$ is an ideal of X, and μ factors through $X \twoheadrightarrow X/\operatorname{Ker}\mu$.*

So we can restrict ourselves to **pure measures**, that is, measures with trivial kernel. For example, the **canonical** measure of an MV-algebra is **pure**.

Definition 8. For an MV-algebra X, let $G_0(X)$ be the group of invertible measures $\mu\colon X \to X$, viewed as a subgroup of $G(X)$. The group $\pi_1(X)$ of $\alpha \in G_0(X)$ with $\mu_X\alpha = \mu_X$ will be called the **fundamental group** of X.

There is a **covering theory** of MV-algebras X for which $\mu_X\colon B(X) \twoheadrightarrow X$ is a universal covering. In particular, we have a canonical representation:

$$\boxed{X \cong B(X)/\pi_1(X)}$$

Coverings of X correspond to subgroups of $\pi_1(X)$.

5. Three types of algebraic logic

MV-algebras formalize Łukasiewicz' **many-valued logic** (Chang 1958), with truth values in the unit interval I. For the "working mathematician", this means that a proposition holds for all MV-algebras if it is valid in the MV-algebra I. We have seen that this type of logic is equivalent to **measure theory** in a wide sense.

Now MV-algebras are commutative L-algebras. So one could expect that **quantum measuring**, usually formalized in terms of operator algebras, is covered by non-commutative L-algebras. This is in fact true, and it does by no means exhaust the ambit of L-algebras.

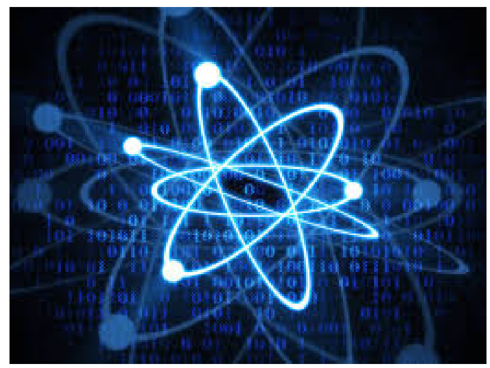
Note that quantum theory has also been found to be a matter of logic. Birkhoff and von Neumann extracted it as the **logic of quantum mechanics** (Ann. Math., 1936). Now classical (Boolean) logic generalizes in three major ways:

logic	models	subject
intuitionistic	locales	**general topology**
Łukasiewicz	MV-algebras	**measure theory**
quantum	orthomodular lattices	**von Neumann algebras**

The models in the table (locales, MV-algebras, and orthomodular lattices) are ***L*-algebras**.

Topology

Measure Theory



Quantum Theory

MV-algebras as generalized measure spaces have already been mentioned. It remains to give a brief description of the L-algebras arising in **topology** and **quantum theory**.

6. Locales

In the standard model of classical logic, propositions are represented by the subsets A of a fixed set X. Negation corresponds to the complement $X \setminus A$.

For intuitionistic logic, X is a topological space, and propositions correspond to open sets U in X. The negation U' of U is given by the largest open set which is disjoint to U, that is, $U' = X \smallsetminus \overline{U}$. In general, double negation leads to a proper inclusion:

$$U \subset U''.$$

Open sets form a complete lattice $\mathscr{O}(X)$ (a **locale**) which determines X in many cases (e. g., if X is Hausdorff). Every locale is a **Brouwerian semilattice**, that is, a $\wedge$-semilattice X with greatest element 1 and an operation $\to$ satisfying

$$\boxed{x \wedge y \leqslant z \iff x \leqslant y \to z}$$

Algebras $(X; \to, 1)$ which embed into a Brouwerian semilattice are called **Hilbert algebras**. Henkin's 1950 theorem states that Hilbert algebras formalize the **deduction theorem**: A proposition x implies y if and only if $x \to y$ is true in any Hilbert algebra. Here is another characterization:

Proposition 9. *A Hilbert algebra is equivalent to an L-algebra which is* **self-distributive***:*

$$\boxed{x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)}$$

Brouwerian semilattices coincide with $\wedge$-closed Hilbert algebras.

So we have inclusions of categories:

$$\mathbf{Top} \subset \mathbf{Loc} \subset \mathbf{BS} \subset \mathbf{Hilb} \subset \mathbf{LAlg}$$

For a topological space X, the map

$$\mathscr{O}(X) \to G(\mathscr{O}(X))$$

into the structure group is given by double negation, a lattice homomorphism (Glivenko's theorem).

7. Quantum logic

Propositions in quantum logic are represented by closed subspaces of a Hilbert space, negation being the orthogonal complement. The closed subspaces form an **orthomodular lattice** (OML), that is,

$$x \leqslant y \implies x \vee (x^{\perp} \wedge y) = y.$$

More generally, the projections of a von Neumann algebra A form an OML.

Proposition 10. *An OML is equivalent to an L-algebra with 0 which satisfies*

$$\boxed{x \cdot 0 \leqslant y \implies y \cdot x = x}$$

Here $x \cdot 0 = x^{\perp}$. Moreover, such an L-algebra is $\wedge$-closed. The lattice operations are given by

$$x \vee y = (x^{\perp} \cdot y^{\perp}) \cdot x, \qquad x \wedge y = (x^{\perp} \vee y^{\perp})^{\perp}.$$

As in the case of MV-algebras, OMLs embed into their structure group:

Theorem 6. *The structure group of an OML X is a right ℓ-group with negative cone $S(X)$. The natural map $X \to G(X)$ embeds X as an interval $[0, 1]$ into $G(X)$, and 0^{-1} is a strong order unit.*

The element 0^{-1} is **singular** in the following sense:

Definition 9. We call an element $s \geqslant 1$ of a right ℓ-group **singular** if $s^{-1} \leqslant xy \Longrightarrow yx = x \wedge y$ holds for all $x, y \leqslant 1$.

Now a **singular strong order unit** of a right ℓ-group is necessarily unique. So we obtain a **group-theoretic** characterization of OMLs (von Neumann algebras, up to duality and trivial factors $M_2(\mathbb{C})$):

Theorem 7. *Up to isomorphism, $X \mapsto G(X)$ is a one-to-one correspondence between OMLs X and right ℓ-groups which admit a singular strong order unit.*

More details: Von Neumann algebras, L-algebras, Baer $*$-monoids, and Garside groups, Forum Math. 30 (2018), no. 4, 973-995