Dual Ore's theorem on distributive intervals of finite groups

Sébastien Palcoux

IMSc (Chennai, India) $\rightarrow$ YMSC (Beijing, China)

Advances in Group Theory and Applications 2019

Wednesday, June 26, 2019

Method to get a dual version of a theorem

- Take a theorem in (finite) group theory,
- extend it to "subfactor planar algebra" (if possible),
- restrict to the dual group case,
- reformulate the statement and the proof in group theory.

It works similarly for getting a relative version.

Contents

Definition

A *lattice* $(L, \wedge, \vee)$ is a partially ordered set (poset) L in which every two elements a, b have a unique infimum (or meet) $a \wedge b$ and a unique supremum (or join) $a \vee b$.

Subgroup lattice

Let G be a group. The set of subgroups $K \subseteq G$ is a lattice, denoted by $\mathcal{L}(G)$, ordered by $\subseteq$, with $K_1 \vee K_2 = \langle K_1, K_2 \rangle$ and $K_1 \wedge K_2 = K_1 \cap K_2$.

Definition

A *sublattice* of $(L, \wedge, \vee)$ is a subset $L' \subseteq L$ such that $(L', \wedge, \vee)$ is also a lattice. Let $a, b \in L$ with $a \leq b$, then the *interval* $[a, b]$ is the sublattice $\{c \in L \mid a \leq c \leq b\}$.

Definition

The lattice $(L, \wedge, \vee)$ is *distributive* if $\forall a, b, c \in L$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

(or equivalently, $\forall a, b, c \in L,\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$)

Theorem (Ore, 1938)

A finite group is cyclic iff its subgroup lattice is distributive.

Relative version of one way (Ore, 1938)

Let $[H, G]$ be a distributive interval in $\mathcal{L}(G)$. Then there is $g \in G$ such that $\langle Hg \rangle = G$ (note that $\langle Hg \rangle = \langle H, g \rangle$).

Converse false: $\langle S_2, (1234) \rangle = S_4$ but $[S_2, S_4]$ is not distributive.

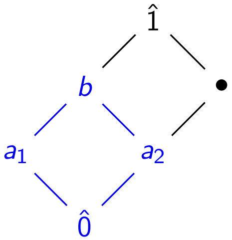
Let $\hat{0}$ and $\hat{1}$ be the *minimum* and *maximum* in a bounded lattice. A distributive bounded lattice is called *Boolean* if every element b admits a unique complement $b^{\complement}$ (i.e. $b \wedge b^{\complement} = \hat{0}$, $b \vee b^{\complement} = \hat{1}$).

Properties

A finite Boolean lattice is isomorphic to the subset lattice of a finite set (whose cardinal is called the rank). A finite lattice is distributive iff it embeds into a Boolean lattice (Birkhoff, 1937).

An *atom* is an element a such that $\forall x \in L,\ a \geq x > \hat{0} \Rightarrow x = a$.

The *bottom interval* of L is $[\hat{0}, b]$ with b the atoms join.



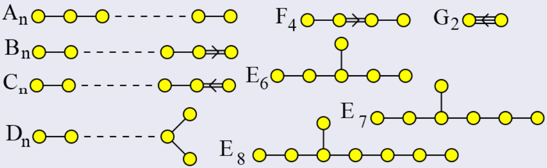
A lattice is called *bottom Boolean* if its bottom interval is Boolean.

Lemma (R.P. Stanley)

A finite distributive lattice is bottom Boolean.

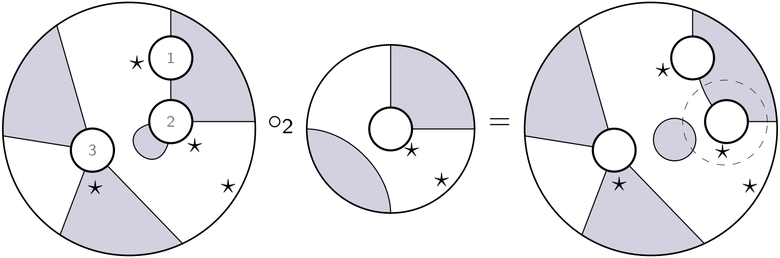
Examples of Boolean intervals "of Lie type"

Let G be a finite Chevalley group and B its Borel subgroup. Then $[B, G]$ is Boolean of rank n, with n the rank of its Dynkin diagram.



The notion of subfactor planar algebra is due to Vaughan Jones (1999). It extends the notion of finite group, but also of finite quantum group (i.e. finite dim. Hopf C*-algebra).

There is no time to define it correctly. In "too much" shortened, it is a "positive" representation of the planar operad. The planar operad is given by compositions of "planar tangles" as follows:



For more details, go to the Wikipedia page "Planar algebra" (the current version was written by myself).

Group-like structures on a subfactor planar algebra (2-box space)

group G	identity projection id
element $g \in G$	minimal (central) projection $u \leq id$
composition gh	co(nvolution)product $u * v$
neutral $eg = ge = g$	trivial projection $e_1 * u = u * e_1 \sim u$
inverse $g^{-1}g = e$	contragredient $\overline{u} * u \succeq e_1$
subgroup $H \subseteq G$	biprojection $b * b \sim \overline{b} = b = b^* = b^2$
subgroup lattice $\mathcal{L}(G)$	biprojection lattice $[e_1, id]$
order $\|G\|$	index $\|id : e_1\| = tr(id)/tr(e_1)$

Surprisingly, the (finite) index can be non-integer!
For a nice class of subfactor planar algebra, called *irreducible*, the biprojection lattice is finite (Watatani).

Ore's theorem on subfactor planar algebra (P.)

An irreducible subfactor planar algebra with a distributive biprojection lattice admits a minimal central projection generating the identity biprojection.

Dual Ore's theorem

Definition

Let W be a representation of a group G, K a subgroup of G, and X a subspace of W. We call the *fixed-point subspace*

$$W^K := \{w \in W \mid kw = w \ , \forall k \in K\}$$

and the *pointwise stabilizer subgroup*

$$G_{(X)} := \{g \in G \mid gx = x \ , \forall x \in X\}$$

Dual Ore's theorem (P., 2018)

Let $[H, G]$ be a distributive interval in $\mathcal{L}(G)$. Then there is V irreducible complex representation of G such that $G_{(V^H)} = H$.

Note that in this dual version, the notion of element $g \in G$ is replaced by the notion of irreducible complex representation.

Idea of the proof

- reduction to the Boolean case via the bottom interval,
- induction on the rank using an induced representation.

Application (P., 2018)

The minimal number of irreducible components for a faithful complex representation of a finite group G is at most the minimal length for an ordered chain of subgroups

$$\{e\} = H_0 < H_1 < \cdots < H = G$$

such that $[H_i, H_{i+1}]$ is distributive (or better, bottom Boolean).

Note that this upper-bound involves the subgroup lattice only, it is a link between combinatorics and representation theory.

If we still have time, we will see in the next section a "possible" improvement of this bound, involving the coset lattice only, related to a conjecture of K.S. Brown.

Dual Euler totient and Brown's conjecture

The *Euler totient* of $[H, G]$ is (with μ the Möbius function)

$$\varphi(H, G) = \sum_{K \in [H,G]} \mu(K, G)|K : H|.$$

It counts the number of cosets Hg such that $\langle Hg \rangle = G$.

Note that $\varphi(\{e\}, C_n) = \varphi(n)$ is the usual Euler's totient function.

The *dual Euler totient* of $[H, G]$ is

$$\hat{\varphi}(H, G) = \sum_{K \in [H,G]} \mu(H, K)|G : K|.$$

Thm (P., 2018): If it is nonzero then $\exists V$ irrep. with $G_{(V^H)} = H$.

As a consequence, the previous upper-bound works if we replace "$[H_i, H_{i+1}]$ distributive" by "$\hat{\varphi}(H_i, H_{i+1})$ nonzero". And this new bound is better if the following open problem has a positive answer.

Problem: $[H, G]$ Boolean implies $\hat{\varphi}(H, G)$ nonzero?

The *Euler characteristic* of $[H, G]$ is

$$\chi(H, G) = - \sum_{K \in [H,G]} \mu(K, G)|G : K|.$$

It is the (reduced) Euler characteristic of the order complex of the coset poset $P = \{Kg \mid g \in G, K \in [H, G)\}$, and also the Möbius invariant of the coset lattice $\overline{P}$.

Relative Brown's conjecture: $\chi(H, G)$ is nonzero (in fact, Brown asked for $H = \{e\}$, after Gaschütz result for G solvable).

But, in the Boolean case we have $\chi(H, G) = \pm\hat{\varphi}(H, G)$.

- Ore's theorem for cyclic subfactor planar algebras and beyond, *Pacific J. Math.* 292-1 (2018), 203-221.
- On Boolean intervals of finite groups (with Mamta Balodi), *J. Combin. Theory Ser. A* 157 (2018), 49-69.
- Ore's theorem on subfactor planar algebras, *Under Review*, arXiv:1704.00745.
- **Dual Ore's theorem on distributive intervals of finite groups**, *J. Algebra*, 505 (2018), 279-287.
- Euler totient of subfactor planar algebras, *Proc. Amer. Math. Soc.* 146 (2018), no. 11, 4775-4786.

Thanks for your attention!