Structure and permutation groups of a solution of the Yang-Baxter equation

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[1] This is a joint work with Adolfo Ballester-Bolinches and Ramon Esteban-Romero

Introduction

- The quantum Yang-Baxter Equation (YBE) is a consistency equation from statistical mechanics.

C. N. YANG: "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction", *Phys. Rev. Lett.* 19 (1967), 1312–1315.

- Open problem: Find all the solutions of the YBE.
- We study a subclass of solutions, the non-degenerate involutive set-theoretic ones.
- This type of solutions is connected with some mathematical topics such as radical rings, trifactorised groups and Hopf algebras.

Definition

A set-theoretic solution of the Yang-Baxter equation is a pair (X, r), where X is a non-empty set and $r\colon X \times X \to X \times X$ is a map such that

$$r_{12}r_{23}r_{12} = r_{23}r_{12}r_{23},$$

where

$$r_{12} = r \times \mathrm{id}_X\colon X \times X \times X \to X \times X \times X,$$
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We denote $r(x, y) = (f_x(y), g_y(x))$ for $x, y \in X$.

- (X, r) is non-degenerate if $f_x, g_x \in \mathrm{Sym}(X)$ for all $x \in X$.
- (X, r) is involutive if $r^2 = r \circ r = \mathrm{id}_{X^2}$.

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Solution of the YBE $\equiv$ Non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.

We can use techniques from group theory to study the solutions of the YBE by considering two fundamental groups associated to a solution (X, r): the structure group and the permutation group.

P. ETINGOF, T. SCHEDLER, A. SOLOVIEV: "Set-theoretical solutions to the quantum Yang-Baxter equation", *Duke Math. J.* 100 (1999), 169–209.

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$$G(X, r) = \langle X \mid xy = f_x(y)g_y(x),\ \text{for } x, y \in X\rangle.$$

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The permutation group of (X, r) is

$$\mathcal{G}(X,r) = \langle f_x \mid x \in X \rangle \leq \mathrm{Sym}(X).$$

Definition

A left brace is a non-empty set B with two binary operations, $+$ and $\cdot$, such that

1. $(B, +)$ is an abelian group,
2. $(B, \cdot)$ is a group,
3. $a \cdot (b + c) = a \cdot b + a \cdot c - a \quad \forall a, b, c \in B$.

If we change condition 3 by

$$(b + c) \cdot a = b \cdot a + c \cdot a - a,$$

then we say that B is a right brace.

A two-sided brace is a left and right brace simultaneously.

W. RUMP: "Braces, radical rings, and the quantum Yang-Baxter equation", *J. Algebra* 307 (2007) 153–170.

Proposition

If (X, r) is a solution of the YBE, then it is possible to define additions in $G(X, r)$ and $\mathcal{G}(X, r)$ in such a way they become left braces.

F. CEDÓ, E. JESPERS, J. OKNIŃSKI: "Braces and the Yang-Baxter equation", *Commun. Math. Phys.* 327 (2014) 101–116.

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Our aim is to use the Cayley graph of $(\mathcal{G}(X, r), \circ)$ to

- Obtain the addition in $\mathcal{G}(X, r)$.
- Describe the left brace structure of $G(X, r)$.

Definition

The Cayley graph of $(\mathcal{G}(X,r),\circ)$ with generating set $\{f_x \mid x \in X\}$ has

- vertices: elements α of $\mathcal{G}(X,r)$,
- edges: $\alpha \xrightarrow{x} \alpha f_x$ for $\alpha \in \mathcal{G}(X,r)$ and $x \in X$.

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Theorem

In the Cayley graph of $(\mathcal{G}(X,r),\circ)$, if we replace the label of the edge $\alpha \xrightarrow{x} \alpha f_x$, $x \in X$, $\alpha \in \mathcal{G}(X,r)$, by $\alpha(x)$, then the labelled graph obtained is the Cayley graph of an abelian group. If $+$ denotes the operation of this group, then $(\mathcal{G}(X,r),+,\circ)$ is a left brace.

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This result says that the addition in $\mathcal{G}(X,r)$ is defined as

$$\alpha + f_{\alpha(x)} = \alpha f_x \text{ for } x \in X, \alpha \in \mathcal{G}(X,r),$$

that is,

$$\alpha + f_z = \alpha f_{\alpha^{-1}(z)} \text{ for } z \in X, \alpha \in \mathcal{G}(X,r).$$

Example

Let (X, r) be the solution of the YBE with $X = \{1, 2, 3, 4, 5\}$ and $f_1 = f_2 = f_3 = 1$, $f_4 = (1, 2)(4, 5)$ and $f_5 = (1, 3)(4, 5)$.

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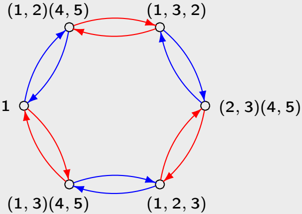

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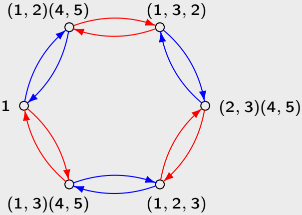

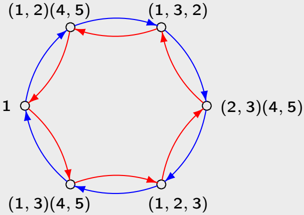

Figure: Cayley graphs of $(\mathcal{G}(X, r), \circ)$ and $(\mathcal{G}(X, r), +)$.

The obtaining of $G(X, r)$

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- $(\mathcal{G}(X,r),\circ)$ acts on on the left on E via $g * e_{\alpha,x} = e_{g\alpha,g(x)}$.
- This action is naturally extended to W and we consider the semidirect product $[W]\mathcal{G}(X,r)$.

The obtaining of $G(X,r)$

- Next we identify all edges with the same label taking the quotient modulo
$$K = \langle e_{\alpha,y} - e_{\beta,y} \mid y \in X, \quad \alpha, \beta \in \mathcal{G}(X,r) \rangle.$$

The obtaining of $G(X,r)$

- Next we identify all edges with the same label taking the quotient modulo
$$K = \langle e_{\alpha,y} - e_{\beta,y} \mid y \in X, \quad \alpha,\beta \in \mathcal{G}(X,r)\rangle.$$

- With this, we can construct the quotient group
$$[W]\mathcal{G}(X,r)/K \cong [W/K]\mathcal{G}(X,r)$$
and take the subgroup
$$H = \langle (\bar{x}, f_x) \mid x \in X\rangle$$
where $\bar{x} = e_{1,x} + K$.

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3. *The product of H has the form*

$$\left(\sum_{x\in X} a_x\bar{x}, \alpha\right)\cdot\left(\sum_{x\in X} b_x\bar{x}, \beta\right) = \left(\sum_{x\in X}(a_x + b_{\alpha^{-1}(x)})\bar{x}, \alpha\beta\right),$$

where $\alpha = \sum_{x\in X} a_x f_x$, $\beta = \sum_{x\in X} b_x f_x$.

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where $\alpha = \sum_{x\in X} a_x f_x$, $\beta = \sum_{x\in X} b_x f_x$.

4. *If we define in H an operation $+$ by means of*

$$\left(\sum_{x\in X} a_x\bar{x}, \alpha\right) + \left(\sum_{x\in X} b_x\bar{x}, \beta\right) = \left(\sum_{x\in X}(a_x + b_x)\bar{x}, \alpha+\beta\right),$$

then $(H,+,\cdot)$ is a left brace.

Example

Let (X, r) be the solution of the YBE with $X = \{1, 2, 3, 4, 5\}$ and $f_1 = f_2 = f_3 = 1$, $f_4 = (1, 2)(4, 5)$ and $f_5 = (1, 3)(4, 5)$.
Let F be the free group on X and $w = 445^{-1} \in F$.

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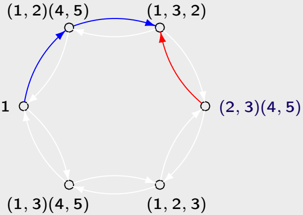

Figure: Path in the Cayley graph of $(\mathcal{G}(X, r), +)$.

Its image in $(\mathcal{G}(X, r), +)$ is: $f_4 + f_4 - f_5 = (2, 3)(4, 5)$.

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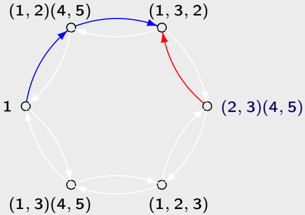

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Its image in $(\mathcal{G}(X, r), +)$ is: $f_4 + f_4 - f_5 = (2, 3)(4, 5)$.
Its image in $(G(X, r), +)$ is:
$(\bar{4}, f_4) + (\bar{4}, f_4) - (\bar{5}, f_5) = (2 \cdot \bar{4} + (-1) \cdot \bar{5}, (2, 3)(4, 5))$.

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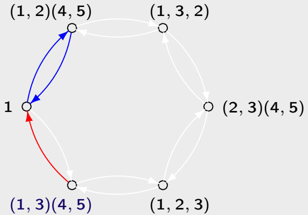

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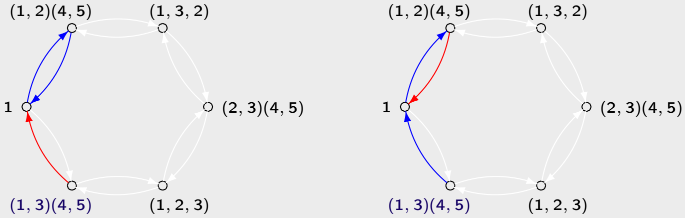

Figure: Paths in the Cayley graphs of $(\mathcal{G}(X, r), \circ)$ and $(\mathcal{G}(X, r), +)$.

Its image in $(G(X, r), \cdot)$ is:
$(\bar{4}, f_4) \cdot (\bar{4}, f_4) \cdot (\bar{5}, f_5)^{-1} = (0 \cdot \bar{4} + 1 \cdot \bar{5}, (1, 3)(4, 5))$.

THANKS FOR YOUR ATTENTION!