# Profinite groups with restricted centralizers of commutators

Marta Morigi

Università di Bologna, Italy

Advances in Group Theory and Applications
25–28 June 2019

Given a group $G$ and an element $x \in G$, we write $x^G$ for the conjugacy class of $x$ in $G$.

Given a group $G$ and an element $x \in G$, we write $x^G$ for the conjugacy class of $x$ in $G$.

A group is said to be a BFC-group if its conjugacy classes are finite and of bounded size.

One of the most famous of B. H. Neumann's theorems (1954) says that in a BFC-group the commutator subgroup $G'$ is finite.
More precisely,

if $|x^G| \leq m$ for each $x \in G$, then the order of $G'$ is bounded by a number depending only on $m$.

The best known bound is $|G'| < m^{(1/2)(7+\log m)}$ by R. M. Guralnick and A. Maroti (2011).

Recently, G. Dierings and P. Shumyatsky obtained a BFC-type theorem for commutators:

**DIERINGS, SHUMYATSKY**

Let $m$ be a positive integer and $G$ a group. If $|x^G| \leq m$ for any commutator $x = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, then $G''$ is finite of $m$-bounded order.
Moreover, If $|x^{G'}| \leq m$ for any commutator $x$, then $\gamma_3(G') = [[G', G'], G']$ has finite $m$-bounded order.

With E. Detomi and P. Shumyatsky, we extended this to arbitrary multilinear commutator words.

A word $w$ on $n$ variables is an element of the free group $F$ with free generators $x_1, \ldots, x_n$.
Given a group $G$, we can think of $w$ as a function $w : G^n \mapsto G$.
We denote by $G_w$ the set of $w$-values and by $w(G)$ the verbal subgroup generated by $G_w$.
When $G$ is a profinite group we always mean “topologically generated”.

A word $w$ on $n$ variables is an element of the free group $F$ with free generators $x_1, \dots, x_n$.
Given a group $G$, we can think of $w$ as a function $w : G^n \mapsto G$.
We denote by $G_w$ the set of $w$-values and by $w(G)$ the verbal subgroup generated by $G_w$.
When $G$ is a profinite group we always mean "topologically generated".

Recall that multilinear commutator words, also known as outer commutator words, are words obtained by nesting commutators but using always different variables.
For example the word $[[x_1, x_2], [x_3, x_4, x_5], x_6]$ is a multilinear commutator word but the word $[x_1, x_2, x_2, x_2]$ is not.

Formally, multilinear commutator words are recursively defined as follows:

**Definition**

The word $w = x_1$ is a multilinear commutator word of weight 1. If $u$, $v$ are multilinear commutator words of weights $m$ and $n$ respectively involving different variables, then $[u, v]$ is a multilinear commutator word of weight $m + n$.

Formally, multilinear commutator words are recursively defined as follows:

### Definition

The word $w = x_1$ is a multilinear commutator word of weight 1. If $u$, $v$ are multilinear commutator words of weights $m$ and $n$ respectively involving different variables, then $[u, v]$ is a multilinear commutator word of weight $m + n$.

### Examples

- the lower central words $\gamma_i$ defined by: $\gamma_1 = x_1$, $\gamma_i = [\gamma_{i-1}, x_i] = [x_1, x_2, \ldots, x_i]$ for $i \geq 1$;
- the derived words $\delta_i$ defined by: $\delta_0 = x_1$, $\delta_i = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}-1}, \ldots, x_{2^i})]$ for $i \geq 1$.

We generalize Neumann's theorem on finiteness of $G'$ where $G$ is a BFC-group, as follows.

### DETOMI, M., SHUMYATSKY

Let $w = w(x_1, \dots, x_k)$ be a multilinear commutator, $m$ an integer and $G$ a group such that $|x^G| \leq m$ for every $x \in G_w$.
Then the commutator subgroup of $w(G)$ has finite $(m, k)$-bounded order.

### DETOMI, M., SHUMYATSKY

Let $w = w(x_1, \dots, x_k)$ be a multilinear commutator, $m$ an integer and $G$ a group such that $|x^{w(G)}| \leq m$ for every $x \in G_w$.
Then $[w(w(G)), w(G)]$ has finite $(m, k)$-bounded order.

This also extend G. Dierings and P. Shumyatsky results on commutators.

Shalev used Neumann's result to investigate profinite groups with restricetd centralizers.

Shalev used Neumann's result to investigate profinite groups with restricetd centralizers.

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. Equivalently, a profinite group is a Hausdorff, compact, and totally disconnected topological group.

Shalev used Neumann's result to investigate profinite groups with restricetd centralizers.

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. Equivalently, a profinite group is a Hausdorff, compact, and totally disconnected topological group.

A group $G$ is said to have restricted centralizers if if for each $g$ in $G$ the centralizer $C_G(g)$ either is finite or has finite index in $G$.

Shalev used Neumann's result to investigate profinite groups with restricetd centralizers.

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. Equivalently, a profinite group is a Hausdorff, compact, and totally disconnected topological group.

A group $G$ is said to have restricted centralizers if if for each $g$ in $G$ the centralizer $C_G(g)$ either is finite or has finite index in $G$.

Groups with restricted centralizers are generalizations of FC-groups.

An element $x \in G$ is an FC-element if $|G : C_G(x)|$ is finite, i.e. if $|x^G|$ is finite.

If $G$ is a group, the set $\Delta(G)$ of FC-elements of $G$ is a subgroup, and it is called the FC-center of $G$.

This happens because $C_G(xy) \geq C_G(x) \cap C_G(y)$ for all $x, y \in G$, so if both $C_G(x)$ and $C_G(y)$ have finite index the same holds for $C_G(xy)$.

Shalev used Neumann's result to investigate profinite groups with restricetd centralizers.

A profinite group is a topological group that is isomorphic to an inverse limit of finite groups. Equivalently, a profinite group is a Hausdorff, compact, and totally disconnected topological group.

A group $G$ is said to have restricted centralizers if if for each $g$ in $G$ the centralizer $C_G(g)$ either is finite or has finite index in $G$.

Groups with restricted centralizers are generalizations of FC-groups.

An element $x \in G$ is an FC-element if $|G : C_G(x)|$ is finite, i.e. if $|x^G|$ is finite.

If $G$ is a group, the set $\Delta(G)$ of FC-elements of $G$ is a subgroup, and it is called the FC-center of $G$.

This happens because $C_G(xy) \geq C_G(x) \cap C_G(y)$ for all $x, y \in G$, so if both $C_G(x)$ and $C_G(y)$ have finite index the same holds for $C_G(xy)$.

A group $G$ is a FC-group if $G = \Delta(G)$.

### SHALEV, 1994

If $G$ is a profinite FC-group then $G'$ is finite, so $G$ is finite-by-abelian.

Here $G'$ denotes the topological closure of the abstract commutator subgroup of $G$.

### Shalev, 1994

If $G$ is a profinite FC-group then $G'$ is finite, so $G$ is finite-by-abelian.

Here $G'$ denotes the topological closure of the abstract commutator subgroup of $G$.

### Remark

A profinite finite-by-abelian group is central-by-finite.

This is because if $T$ is a normal finite subgroup of $G$ such that $G/T$ is abelian, then $G' \leq T$. As $G$ is profinite, there exists an open normal subgroup $N$ of $G$ such that $T \cap N = 1$.
Then $[N, G] \leq G' \cap N \leq T \cap N = 1$, so $N$ is contained in the center of $G$.

Shalev's result is actually more general:

**SHALEV, 1994**

A profinite group with restricetd centralizers is abelian-by-finite. More precisely: the (abstract) subgroup $\Delta(G)$ is closed in $G$, it has finite index in $G$ and its commutator subroup is finite.

So $G$ is finite-by-abelian-by-finite and thus abelian-by-finite.

We obtained a “verbal” version of the result by Shalev.

**THEOREM (DETOMI, M., SHUMYATSKY)**

Let $w$ be a multilinear commutator word and $G$ a profinite group in which all centralizers of $w$-values are either finite or of finite index. Then $w(G)$ is abelian-by-finite.

The proof of the above result requires some combinatorial techniques for handling multilinear commutators which were introduced by Fernández-Alcober an M. and then developed with Detomi and Shumyatsky.

## Proposition

Let $A_1, \dots, A_n$ be normal subgroups of a profinite group $G$. Define $\mathcal{X}_w(A_1, \dots, A_n) = \{w(g_1, \dots, g_n) | g_i \in A_i \text{ for all } i\}$.
Let $H$ be the topological closure of the abstract subgroup $\Delta(G)$.
If $\mathcal{X}_w(A_1, \dots, A_n) \subseteq \Delta(G)$ then $[H, w(A_1, \dots, A_n)]$ is finte.

Proof of the Theorem.

Suppose $x$ is a $w$-value with infinite order. Then $C_G(x)$ is infinite, hence of finite index.
Let $N$ be an open normal subgroup of $G$ contained in $C_G(x)$, and let $K = K_1 \cdots K_n$, where $K_i = w(G, \dots, N, \dots, G)$ (here $N$ appears in the $i$-th entry).
Let $y = w(g_1 \dots, u, \dots, g_n)$, where $u \in N$ appears in the $i$-th entry.
Then $y \in N \leq C_G(x)$, so $x$ centralizes $y$ and thus $C_G(y)$ is infinite, so $y \in \Delta(G)$.
It follows from the above proposition that $[H, K_i]$ is finite, thus $[H, K] = \prod_i [H, K_i]$ is finite.
As $K \leq H$ we have that $K'$ is finite.

Moreover, from the fact that $N$ is open in $G$ it follows that $K$ is open in $w(G)$.
So $w(G)$ is finite-by-abelian-by-finite and thus abelian-by-finite.

Moreover, from the fact that $N$ is open in $G$ it follows that $K$ is open in $w(G)$.
So $w(G)$ is finite-by-abelian-by-finite and thus abelian-by-finite.

Hence, we can assume that all $w$-values in $G$ have finite order.

Moreover, from the fact that $N$ is open in $G$ it follows that $K$ is open in $w(G)$.
So $w(G)$ is finite-by-abelian-by-finite and thus abelian-by-finite.

Hence, we can assume that all $w$-values in $G$ have finite order.

**REMARK**

If all $w$-values are FC-elements we argue as above with $N = G$ and we get that $w(G)'$ is finite.

Indeed, in this case $K = w(G)$.

It can be proved that when all $w$-values in $G$ have finite order then $w(G)$ is locally finite.

It can be proved that when all $w$-values in $G$ have finite order then $w(G)$ is locally finite.

Now we can apply Wilson's theorem (1983) on the structure of compact torsion groups which implies that in our situation $w(G)$ has a finite series of normal subgroups such that each factor either is a pro-$p$-group or is isomorphic to a Cartesian product of finite nonabelian simple groups.

It can be proved that when all $w$-values in $G$ have finite order then $w(G)$ is locally finite.

Now we can apply Wilson's theorem (1983) on the structure of compact torsion groups which implies that in our situation $w(G)$ has a finite series of normal subgroups such that each factor either is a pro-$p$-group or is isomorphic to a Cartesian product of finite nonabelian simple groups.

- It follows from Ore conjecture (Liebeck, O'Brien, Shalev, Tiep, 2010) that in a cartesian product $U$ of finite simple groups every element is a commutator, thus every element is a $w$-value. Thus we are in a condition where the centralizer of each element is either finite or of finite index and wh apply Shalev's result. So $U$ is abelian-by-finite, thus finite.

It can be proved that when all $w$-values in $G$ have finite order then $w(G)$ is locally finite.

Now we can apply Wilson's theorem (1983) on the structure of compact torsion groups which implies that in our situation $w(G)$ has a finite series of normal subgroups such that each factor either is a pro-$p$-group or is isomorphic to a Cartesian product of finite nonabelian simple groups.

- It follows from Ore conjecture (Liebeck, O'Brien, Shalev, Tiep, 2010) that in a cartesian product $U$ of finite simple groups every element is a commutator, thus every element is a $w$-value. Thus we are in a condition where the centralizer of each element is either finite or of finite index and wh apply Shalev's result. So $U$ is abelian-by-finite, thus finite.

- For dealing with the pro-$p$ factors we rely on the techniques developed by Zelmanov for the solution of the restricetd Burnside Problem.

## Corollary 1

Let $w$ be a multilinear commutator word and $G$ a profinite group in which all centralizers of $w$-values are either finite or of finite index. Then $G$ is soluble-by-finite.

### Corollary 1

Let $w$ be a multilinear commutator word and $G$ a profinite group in which all centralizers of $w$-values are either finite or of finite index. Then $G$ is soluble-by-finite.

### Corollary 2

If $w$ is a multilinear commutator word and $G$ a profinite group in which every nontrivial $w$-value has finite centralizer, then either $w(G) = 1$ or $G$ is finite.

### Corollary 1

Let $w$ be a multilinear commutator word and $G$ a profinite group in which all centralizers of $w$-values are either finite or of finite index. Then $G$ is soluble-by-finite.

### Corollary 2

If $w$ is a multilinear commutator word and $G$ a profinite group in which every nontrivial $w$-value has finite centralizer, then either $w(G) = 1$ or $G$ is finite.

We use

Let $w$ be a multilinear commutator word and $G$ a group. If $H$ is a normal subgroup of $G$ such that $N \cap G_w = 1$ then $N$ centralizes $w(G)$.

Naturally, a corresponding result for finite groups must be of quantitative nature:

Let $m$ be a positive integer, $w$ a group-word, and $G$ a finite group such that $w(G) \neq 1$ and $C_G(x)$ has order at most $m$ for each nontrivial $w$-value $x$ of $G$. Does it follows that the order of $G$ is bounded in terms of $m$ and $w$ only?

Naturally, a corresponding result for finite groups must be of quantitative nature:

Let $m$ be a positive integer, $w$ a group-word, and $G$ a finite group such that $w(G) \neq 1$ and $C_G(x)$ has order at most $m$ for each nontrivial $w$-value $x$ of $G$. Does it follows that the order of $G$ is bounded in terms of $m$ and $w$ only?

It is not difficult to see that for some words $w$ the answer is negative. In particular, this happens when $w = x^n$ is a power word, with $n > 1$.

## Example

Let $w = x^3$ and let $N$ be an elementary abelian 3 group and let $a$ be an involution acting on $N$ by inverting all elements. Then in the semidirect product $G = N \rtimes \langle a \rangle$ all elements ouside $N$ are involutions and they are self-centralizing. The set of nontrivial $w$-values is precisely $G \setminus N$ and all such elements have a centralizer of order $m = 2$, while the order of $G$ can be arbitrarily large.

### Example

Let $w = x^3$ and let $N$ be an elementary abelian 3 group and let $a$ be an involution acting on $N$ by inverting all elements. Then in the semidirect product $G = N \rtimes \langle a \rangle$ all elements ouside $N$ are involutions and they are self-centralizing. The set of nontrivial $w$-values is precisely $G \setminus N$ and all such elements have a centralizer of order $m = 2$, while the order of $G$ can be arbitrarily large.

On the other hand, if all nontrivial elements of a finite group $G$ have centralizers of order at most $m$, then $|G|$ is $m$-bounded.

### Isaacs, 1986

If $G$ is a soluble group where every nontrivial element has a centralizer of order at most $m$, then $G$ has order at most $m^2$.

So, the answer is positive for $w = x$.

## Detomi, M., Shumyatsky

Let $p$ be a prime, $q_1, \dots, q_n$ some $p$-powers and $v = v(x_1, \dots, x_n)$ a multilinear commutator word of weight at least 2.
Set $w = v(x_1^{q_1}, \dots, x_n^{q_n})$.
Assume that $G$ is a finite group such that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial $w$-value $x$ of $G$.
Then the order of $G$ is $(w, m)$-bounded.

### Detomi, M., Shumyatsky

Let $p$ be a prime, $q_1, \dots, q_n$ some $p$-powers and $v = v(x_1, \dots, x_n)$ a multilinear commutator word of weight at least 2.
Set $w = v(x_1^{q_1}, \dots, x_n^{q_n})$.
Assume that $G$ is a finite group such that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial $w$-value $x$ of $G$.
Then the order of $G$ is $(w, m)$-bounded.

The proof uses the following result, due to Hartley, which depends on the classification of finite simple groups.

### Hartley 1992

There exists an integer-valued function $f(m)$ such that if $G$ is a finite group containing an element $x$ with $|C_G(x)| \leq m$, then $G$ has a soluble normal subgroup of index at most $f(m)$.

In the case where $w$ is a multilinear commutator word, the result follows easily from Hartley's theorem.
Indeed let $T$ be a soluble characteristic subgroup of $G$ of bounded index. Let $i$ be the smallest integer such that $T^{(i)} \cap G_w = 1$. Then $T^{(i)}$ centralizes $w(G)$ and so $T^{(i)}$ has order at most $|C_G(w(G))| \leq m$.
Pass to the quotient over $T^{(i)}$.
Now $A = T^{(i-1)} \lhd G$ is abelian and there exists $x \in A \cap G_w \neq 1$, so both $A$ and $C_G(A)$ have order at most $|C_G(x)| \leq m$ and so $|G| = |C_G(A)||G/C_G(A)| \leq mm!$.

In the case where $w$ is a multilinear commutator word, the result follows easily from Hartley's theorem.
Indeed let $T$ be a soluble characteristic subgroup of $G$ of bounded index. Let $i$ be the smallest integer such that $T^{(i)} \cap G_w = 1$. Then $T^{(i)}$ centralizes $w(G)$ and so $T^{(i)}$ has order at most $|C_G(w(G))| \leq m$.
Pass to the quotient over $T^{(i)}$.
Now $A = T^{(i-1)} \lhd G$ is abelian and there exists $x \in A \cap G_w \neq 1$, so both $A$ and $C_G(A)$ have order at most $|C_G(x)| \leq m$ and so $|G| = |C_G(A)||G/C_G(A)| \leq mm!$.

For $w = v(x_1^{q_1}, \ldots, x_n^{q_n})$, where $q_i$ are $p$-powers, we use a result by Khukhro: if $P$ is a finite $p$-group admitting a $p$-automorphism of order $p^s$ with $p^m$ fixed points, then $P$ has a characteristic $(p, s, m)$-bounded-index soluble subgroup of $(p, s)$-bounded derived length.

Recall that the $n$th Engel word is defined inductively by $[x,_1 y] = [x, y]$, and $[x,_n y] = [[x,_{n-1} y], y]$.

**DETOMI, M., SHUMYATSKY**

Let $w$ be the $n$th Engel word or the word $w = [x^k,_n y]$, with $n, k \geq 1$. Assume that $G$ is a finite group such that $w(G) \neq 1$ and $|C_G(x)| \leq m$ for every nontrivial $w$-value $x$ of $G$.
Then the order of $G$ is $(w, m)$-bounded.

Here we use previous results on how the exponents of Sylow $p$-subgroups of $w(G)$ can bound the exponent of $w(G)$.