Minimal Embeddings of Small Finite Groups

Lecce 2019

(Joint work with R. Heffernan and D. MacHale)

Embeddings

- **Definition** The group G is *embedded* in the group K if G is isomorphic to a subgroup of K.

- **Example** Cayley's Theorem yields an embedding of any group of order n into $K = S_n$.

- Often, the 'target group' K is specified in advance and the aim is to study the embedded group G.

Slightly different approach

- Given a collection of finite groups $G_1, \ldots, G_r$, what can we say about a group K in which all these groups can be embedded?

- In particular, what can we say about the *order* of such a group K?

‘Natural’ questions (MacHale)

- **Question 1** What is the minimal order of a group K in which all groups of order n can be embedded? When is K unique?

- **Question 2** What is the minimal order of a group K in which all groups of order n <u>or less</u> can be embedded?

- We focus on Question 1 for $n \leq 15$.

Groups of order n for $n \leq 15$

- Abelian or dihedral apart from $n = 8$ and $n = 12$.
- For $n = 8$, the groups of order n are: C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_4, and Q_2
- For $n = 12$, the groups of order n are: C_{12}, $C_2 \times C_2 \times C_3$, D_6, Q_3 and A_4

Elementary bounds

By Lagrange's and Cayley's theorems, if K is a group of minimal order in which all groups of order n can be embedded, then:

$$n \leq |K| \leq n!$$

Lower bounds for *p*-groups

Theorem 1 Let p be a prime and let K be a group of minimal order in which all groups of order p^s can be embedded. Then $|K|$ is a multiple of p^{2s-1}.

Theorem 2 Let p be an <u>odd</u> prime and let $s \geq 3$. Let K be a group of minimal order in which all groups of order p^s can be embedded. Then $|K|$ is a multiple of p^{2s}.

The case $n = 8$

Theorem 3 The groups of minimal order in which all groups of order 8 can be embedded are:

(i) $\langle x, y \mid x^8 = y^2 = 1, yxy = x^3 \rangle \times C_2$

(ii) $C_8 \rtimes \mathrm{Aut}(C_8)$.

The case $n = 12$

Theorem 4 There is a unique group of minimal order in which all groups of order 12 can be embedded, namely $S_3 \times S_4$.

Proof of Theorem 4

- Sylow 2-subgroups of *K* have order at least 8.
- Sylow 3-subgroups of *K* are non-cyclic ⇨ $|K|$ is a multiple of 72.
- But 72 is too small!
- All groups of order 12 <u>can</u> be embedded in $S_3 \times S_4$.
- All groups of order 12 cannot be embedded in any other group of order 144.
- 'Pen & paper' or GAP.

Why stop at *n* = 15?

n - number of groups of order *n*

16	-	14
32	-	51
64	-	207
128	-	2 328
256	-	56 092
512	-	10 494 213
1 024	-	49 487 365 422

(Besche, Eick, O’Brien 2001)

n = 16 - The story so far

There exists a group of order $512 = 2^9$ in which all groups of order 16 can be embedded.

Conjectures

- **Conjecture 1** | S_n | is not minimal with respect to the embedding of all groups of order n for $n \geq 2$.

- **Conjecture 2** | S_n | is not minimal with respect to the embedding of all groups of order n or less for $n \geq 6$.

An Advertisement

MUNSTER GROUPS 2019

Saturday 7th September 2019

Venue: Room F04, Cork Road Campus,

Waterford Institute of Technology

PROVISIONAL PROGRAMME

10.00 **Welcome and Registration**

10.20 **Paul Barry, WIT** *The Riordan Group*

11.10 **Break**

11.30 **Des MacHale, UCC** *Are There More Finite Rings than Finite Groups?*

12.00 **Rex Dark, NUIG** *The smallest nontrivial complete groups of odd order*

12.50 **Lunch**

13.50 **J.P. McCarthy, CIT** *The Ergodic Theorem for Random Walks: from Finite Groups, to Group Algebras, to Finite Quantum Groups*

14.40 **Short Talk 2**

15.10 **Short Talk 3**

15.40 **Break**

16.00 **Ted Hurley, NUIG** *From groups to group rings to codes and information*

16.50 **End**