Set-theoretical solutions of the pentagon equation on groups

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Solutions of the pentagon equation on a vector space

Definition

Let V be a vector space over a field K. A linear operator $S \in End(V \otimes V)$ is called a *solution of the pentagon equation (PE)* if it satisfies

$$S_{12}S_{13}S_{23} = S_{23}S_{12},$$

where $S_{12} = S \otimes \mathrm{id}_V$, $S_{23} = \mathrm{id}_V \otimes S$, $S_{13} = (\tau \otimes \mathrm{id}_V)S_{12}(\tau \otimes \mathrm{id}_V)$ (here τ denotes the flip map $v \otimes w \to w \otimes v$).

Set-theoretical solutions of the pentagon equation

Definition

A *set-theoretical solution of the pentagon equation (PE)* on an arbitrary set M is a map $s : M \times M \to M \times M$ which satisfies the "reversed" pentagon relation

$$s_{23}\, s_{13}\, s_{12} = s_{12}\, s_{23},$$

where $s_{12} = s \times \mathrm{id}_M$, $s_{23} = \mathrm{id}_M \times s$ and $s_{13} = (\mathrm{id}_M \times \tau)\, s_{12}\, (\mathrm{id}_M \times \tau)$ (here τ denotes the flip map $(x, y) \to (y, x)$).

Link between the two equations

Let K be a field, M a finite set, and $V := K^M$. Then, $V \otimes V \cong K^{M\times M}$.

For each map $s : M \times M \to M \times M$ one can associate its pull-back S, i.e., the linear operator $S \in End(V \otimes V)$ such that

$$(S\varphi)(x, y) = \varphi(s(x, y)), \quad \varphi \in K^{M\times M}.$$

S is a solution of the PE if and only if s is a set-theoretical solution of the PE.

Hereinafter, we briefly call a set-theoretical solution s a *solution*.

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A pioneering work

For a map $s : M \times M \to M \times M$ define binary operations $\cdot$ and $*$ via

$$s(x,y) = (x \cdot y\ ,\ x * y),$$

for all $x, y \in M$.

Proposition (Kashaev-Sergeev, 1998)

The map s is a solution on M if and only if the following conditions hold

1. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
2. $(x * y) \cdot ((x \cdot y) * z) = x * (y \cdot z)$,
3. $(x * y) * ((x \cdot y) * z) = y * z$,

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Examples of solutions

1. *(Militaru solutions)* If M is a set, f, g are maps from M into itself such that $f^2 = f$, $g^2 = g$, and $fg = gf$, then
$$s(x, y) = (f(x), g(y))$$
is a solution on M.

2. If $(M, \cdot)$ is a semigroup, $\gamma \in \mathrm{End}(M)$, $\gamma^2 = \gamma$, the map
$$s(x, y) = (x \cdot y, \gamma(y))$$
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3. *(Kac-Takesaki solutions)* If $(M, \cdot)$ is a group, the maps
$$s(x, y) = (x \cdot y, y) \quad \text{and} \quad t(x, y) = (x, x^{-1} \cdot y)$$
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Examples of solutions on factorizable groups

Let M be a group and A, B two subgroups such that $A \cap B = \{1\}$ and $M = AB$. Let $p_1 : M \to A$ and $p_2 : M \to B$ be maps such that $x = p_1(x)\, p_2(x)$, for every $x \in M$.

4. (*Zakrzewski solution*) The map

$$s(x, y) = \left(p_2\left(yp_1(x)^{-1}\right)x,\ yp_1(x)^{-1}\right)$$

is solution on M.

5. (*Baaj-Skandalis solution*) The map

$$s(x, y) = \left(xp_1\left(p_2(x)^{-1}y\right),\ p_2(x)^{-1}y\right)$$

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Kashaev and Sergeev proved that if $(M, \cdot)$ is a group, then the only **invertible** solution $s(x, y) = (x \cdot y, x * y)$ on M is given by

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A new notation

If $s(x, y) = (x \cdot y, x * y)$ is a solution, we set $x * y =: \theta_x(y)$, for all $x, y \in M$, where $\theta_x : M \to M$ is a map, for every $x \in M$.

Proposition

The map $s(x, y) = (x \cdot y,\ \theta_x(y))$ is a solution on a set M if and only if

1. $(M, \cdot)$ *is a semigroup,*
2. $\theta_x(y \cdot z) = \theta_x(y) \cdot \theta_{x \cdot y}(z)$,
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hold, for all $x, y, z \in M$.

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The kernel of a solution

Although θ_1 is not a homomorphism, we have the following result.

Proposition (Catino, Miccoli, M., 2019)

Let $s(x, y) = (x \cdot y, \theta_x(y))$ be a solution on a group $(M, \cdot)$. The subset of M

$$K := \{x \mid x \in M,\ \theta_1(x) = 1\},$$

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Description of solutions on groups

Theorem (Catino, Miccoli, M., 2019)

Let $(M, \cdot)$ be a group and $K \trianglelefteq M$. Moreover, consider

- *R a system of representatives of M/K such that $1 \in R$,*
- *$\mu : M \to R$ a map such that $\mu(x) \in K \cdot x$, for every $x \in M$.*

Then, the map $s : M \times M \to M \times M$ given by

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Conversely, if $s(x, y) = (x \cdot y, \theta_x(y))$ is a solution on M, for all $x, y \in M$, there exists $K \trianglelefteq M$, the kernel of s, such that

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Applications

1. The unique invertible solution on a group $(M, \cdot)$ is given by
$$s(x, y) = (x \cdot y, y).$$

2. Let $n \geq 3$ and $\mathcal{S}_n$ the symmetric group of order n. Consider
   - $K = \mathcal{A}_n$,
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Solutions on groups $(M, *)$

Another question is to describe all the solutions $s(x, y) = (x \cdot y, x * y)$ when $(M, *)$ is a group.

Proposition (Catino, Miccoli, M., 2019)

Let $(M, *)$ be a group. Then, $s(x, y) = (x \cdot y, x * y)$ is a solution on M if and only if

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In order to obtain new solutions, we focus on constructions of solutions on the Cartesian product of two semigroups S and T [Catino, Stefanelli, M., in preparation].

From now on, we set $x \cdot y =: xy$.

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How to find new examples of solutions on semigroups $(S, \cdot)$?

In order to obtain new solutions, we focus on constructions of solutions on the Cartesian product of two semigroups S and T [Catino, Stefanelli, M., in preparation].

From now on, we set $x \cdot y =: xy$.

Matched product of solutions - I

Let S, T be semigroups, $s(a,b) = (ab, \theta_a(b))$ and $t(u,v) = (uv, \theta_u(v))$ solutions on S and T, respectively. Let $\alpha : T \to S^S$ and $\beta : S \to T^T$ be two maps, and set

$$\forall u \in T \quad \alpha_u := \alpha\left(u\right), \qquad \forall a \in S \quad \beta_a := \beta\left(a\right).$$

If the following conditions are satisfied

$$\begin{aligned}
&\alpha_v\left(a\alpha_u\left(b\right)\right) = \alpha_v\left(a\right)\alpha_{\beta_a(v)u}\left(b\right),\\
&\beta_c\left(\beta_b\left(u\right)v\right) = \beta_{b\alpha_v(c)}\left(u\right)\beta_c\left(v\right),\\
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\end{aligned}$$

for all $a, b, c \in S$ and $u, v \in T$, then we call (s, t, α, β) a *matched quadruple*.

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Matched product of solutions - II

The first two conditions

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ensure that $S \times T$ endowed with the operation defined by

$$(a,u)(b,v) = (a\alpha_u(b), \beta_b(u)v),$$

is a semigroup, called the *matched product of S and T*, and we denote it by $S \bowtie T$.

Theorem (Catino, Stefanelli, M. , work in progress)

Let S and T be two semigroups, and (s,t,α,β) a matched quadruple. Then, the map $s \bowtie t : (S\times T)^2 \to (S \times T)^2$ defined by

$$s \bowtie t\,(a,u\,;\,b,v) = \left(a\alpha_u(b), \beta_b(u)v\,;\,\theta_a\alpha_u(b), \theta_{\beta_b(u)}(v)\right),$$

for all $(a,u), (b,v) \in S \times T$, is a solution on $S \bowtie T$.

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Remark

In addition, if S and T are monoids with identity 1_S and 1_T respectively, we have to require the following conditions

$$\begin{aligned} &\alpha_{1_T} = \mathrm{id}_S, \\ &\beta_{1_S} = \mathrm{id}_T, \\ \forall\ a \in S \quad &\beta_a(1_T) = 1_T, \\ \forall\ u \in T \quad &\alpha_u(1_S) = 1_S, \end{aligned}$$

so that $S \bowtie T$ is a monoid with identity $(1_S, 1_T)$.

In this case, if (s, t, α, β) is a matched quadruple, conditions become easier:

$$\begin{aligned} &\theta_a = \alpha_{\theta_u(v)}\theta_a = \theta_{\alpha_v(a)}\alpha_{\beta_a(v)}, \\ &\beta_{\theta_a\alpha_{uv}(b)}\theta_u(v) = \theta_{\beta_{\alpha_v(b)}(u)}\beta_b(v). \end{aligned}$$

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An example

Let us consider

- S a semigroup, $\gamma \in End(S)$, $\gamma^2 = \gamma$, and $s(a, b) = (ab, \gamma(b))$ a solution on S;
- T a semigroup and $t(u, v) = (uv, v)$ a solution on T;
- $\alpha_u = \gamma$, for every $u \in T$;
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Work in progress

A link between the pentagon equation
and the Yang-Baxter equation

Tomorrow at Paola Stefanelli's talk

Thanks for your attention!