A Random Walk through the Theory of Soluble Groups

A talk prepared for AGTA 2019 Lecce

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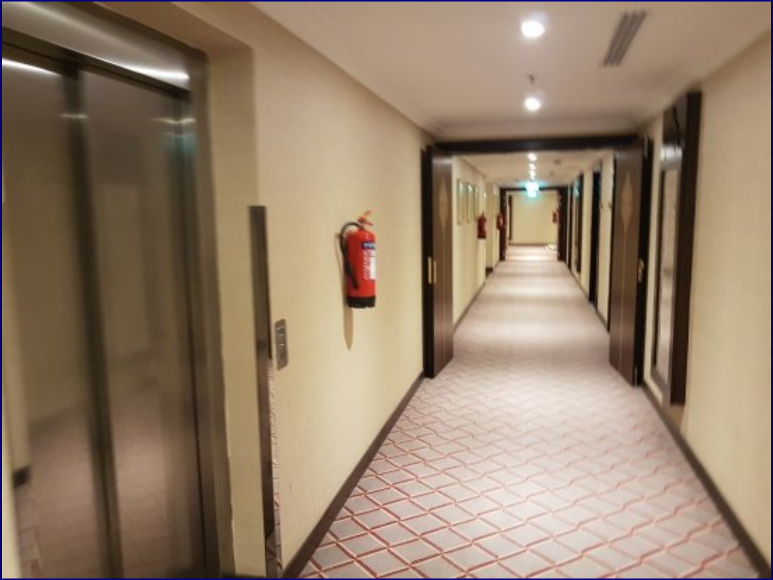
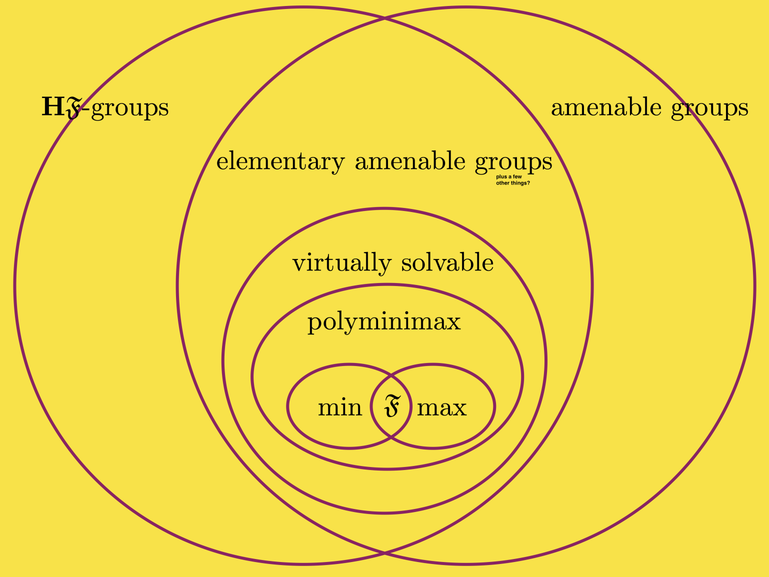
corridor



classes of groups



quasicyclic groups

For a prime p the *quasicyclic group* C_{p^∞} is the group of p-power roots of unity in $\mathbb{C}$. The exponential map $\mathbb{C} \to \mathbb{C}^\times$ given by $z \mapsto e^{2\pi i z}$ is a surjective group homomorphism with kernel $\mathbb{Z}$. By restricting to the subring $\mathbb{Z}\left[\frac{1}{p}\right]$ we can view quasicylic groups additively as well as multiplicatively.

ring	**s.e.s.**	**isomorphism**
$\mathbb{C}$	$\mathbb{Z} \rightarrowtail \mathbb{C} \twoheadrightarrow \mathbb{C}^\times$	$\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^\times$
$\mathbb{R}$	$\mathbb{Z} \rightarrowtail \mathbb{R} \twoheadrightarrow S^1$	$\mathbb{R}/\mathbb{Z} \cong S^1$
$\mathbb{Z}\left[\frac{1}{p}\right]$	$\mathbb{Z} \rightarrowtail \mathbb{Z}\left[\frac{1}{p}\right] \twoheadrightarrow C_{p^\infty}$	$\mathbb{Z}\left[\frac{1}{p}\right] \Big/ \mathbb{Z} \cong C_{p^\infty}$

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- The virtually soluble groups with min are finite extensions of direct products of finitely many quasicyclic groups: they are called Černikov groups.
- The virtually soluble groups with max are polycyclic-by-finite.

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- B is finitely generated (and so has the maximal condition on subgroups)
- A/B has the minimal condition on subgroups (and so is a direct sum of finitely many quasicylic groups and a finite group).

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Example

If $\alpha \in \mathbb{C}$ is an algebraic number then $\mathbb{Z}[\alpha]^+$ is minimax.

Example (...an explicit example ...)

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Example (. . . an explicit example . . .)

$$\mathbb{Z}^2 \rightarrowtail \mathbb{Z}\left[\frac{1}{4+7i}\right]^+ \twoheadrightarrow C_{5^\infty} \oplus C_{13^\infty}.$$

polyminimax groups

I define a group G to be *polyminimax* if there is a series

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in which the factors G_i/G_{i-1} are cyclic or quasicyclic or finite.

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... and with Derek Robinson's generally accepted terminology "minimax group".

In this lecture:

polyminimax means *polyminimax in the sense of Kropholler.*

Minimax Fire Extinguisher

art deco gate of the original minimax factory

Hirsch length etc.

Let G be a polyminimax group and let

$$1 = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G \qquad \ddagger$$

be a cyclic–quasicyclic–finite series that is witness to that.
The following are invariants of G, independent of the choice of $\ddagger$.

- The Hirsch length $h(G)$ counts the number of infinite cyclic factors in $\ddagger$.

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A *π-polyminimax group* is a polyminimax group G such that $\pi(G) \subseteq \pi$.

Baer's original definition

From Baer's paper

Polyminimaxgruppen, Math. Annalen 175, 1–43 (1968).

Besitzt eine abelsche Gruppe A eine Untergruppe U, von deren Untergruppen die Maximalbedingung und von deren Obergruppen die Minimalbedingung erfüllt wird, so heißt A Minimaxgruppe. ...

Besitzt eine Gruppe G eine endliche Kette von Untergruppen N_i mit den Eigenshaften:

$1 = N_0$, N_i *ist ein Normalteiler von* N_{i+1} *mit [abelscher] Minimaxfaktorgruppe* N_{i+1}/N_i, $N_n = G$,

so heiße G Polyminimaxgruppe; ...

Notice that Baer emphasises the restriction to abelian factor groups. Robinson employed a much more general use for the term *minimax*, including all group that are *poly(max or min)*.

the alternative definitions

Theorem

Let G be a group. Then the following are equivalent.

1. *G is a polyminimax group in Kropholler's sense.*
2. *G is virtually polyminimax in Baer's sense.*
3. *G is virtually soluble minimax in Robinson's sense.*

Theorem (An Instance of the Tits Alternative, Tits 1972)

Let G be a finitely generated subgroup of $\mathrm{GL}_n(\mathbb{Q})$. Then either G is polyminimax or G has a subgroup that is free on 2 generators.

Upper triangular matrices provide a source of polyminimax groups.

$$\begin{pmatrix} * & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{pmatrix}$$

If π is a set of primes and n is a π-number then the group of upper triangular matrices with entries from $\mathbb{Z}\left[\frac{1}{n}\right]$ is π-polyminimax. It is also nilpotent-by-abelian.

Theorem

All polyminimax groups are nilpotent-by-abelian-by-finite.

Abel's Groups

For each prime p, Herbert Abels' sequence of p-polyminimax groups:

$$H_n = \left\{ \begin{pmatrix} 1 & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+2}\left(\mathbb{Z}\left[\frac{1}{p}\right]\right) \right\}$$

Theorem

H_n is of type FP_n but not of type FP_{n+1}.

Abels' groups have centre isomorphic to $\mathbb{Z}\left[\frac{1}{p}\right]$ and so can be used to construct groups of type FP_n with a quasicyclic subgroup for arbitrarily large n.

For each prime p the standard restricted wreath product

$$L_p := \mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}$$

is known as a *lamplighter group.*

Theorem (The Kropholler Alternative, Kropholler 1984)

Let G be a finitely generated soluble group. Then G is polyminimax if and only if G has no lamplighter sections.

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“Minimax extinguishes the lamplighter” — Yves de Cornulier, 2016.

bounded generation

Theorem (Kropholler 1984)

Finitely generated polyminimax groups are boundedly generated.

Corollary

If $A \rightarrowtail G \twoheadrightarrow Q$ is a group extension in which G is finitely generated, Q is polyminimax, and A is abelian, and if G has no lamplighter sections, then for any field k, the kQ-module $A \otimes k$ is locally finite dimensional.

To establish Kropholler's Alternative it becomes important to understand how the cohomology functor $H^2(Q, \quad)$ commutes or fails to commute with direct limit (filtered colimits). In general, finitely generated polyminimax groups need not be of type FP_2.

Open Question

Are all boundedly generated soluble groups necessarily polyminimax?

Theorem (Kropholler 1986; K–Martínez-Pérez–Nucinkis 2009)

Let G be a virtually soluble group. Then the following are equivalent:

1. $\mathrm{hd}(G) = \mathrm{cd}(G) < \infty$.
2. *G is of type* FP.
3. *G is a duality group.*
4. *G is torsion-free and constructible.*
5. *G admits a finite classiying space for proper actions.*

Theorem (Kropholler 1993; K–Martínez-Pérez–Nucinkis 2009)

Let G be an elemenatary amenable group of type FP_∞. Then G is polyminimax, virtually torsion-free, and admits a cocompact classifying space for proper actions.

Theorem (Stammbach, 1970)

Let k be a field of characteristic zero and let G be a soluble group. Then $\mathrm{hd}_k(G) = h(G)$.

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Using Stammbach's methods it can be shown that for an arbitrary non-zero commutative ring k and soluble group G then the following are equivalent:

- $\mathrm{hd}_k(G) < \infty$.
- $h(G) < \infty$ and G has no k-torsion.

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Theorem (Kropholler–Martínez-Pérez, 2021)

Let k be a non-zero commutative ring and let G be a soluble group. If $\mathrm{hd}_k(G) < \infty$ then $\mathrm{hd}_k(G) = h(G)$.

residual finiteness

Theorem

Let G be a polyminimax group. Then the finite residual of G is a direct sum of finitely many quasicyclic groups.

Theorem

Let G be a polyminimax group. Then the following are equivalent:

1. *G is residually finite.*
2. *G is virtually torsion-free.*
3. *G has no quasicyclic subgroups.*
4. *G is $\mathbb{Q}$-linear.*

finitely supported probability density functions on a group

Let G be a group. A finitely supported probability density function μ on G is an element of the real group ring $\mathbb{R}G$ whose coefficients lie in the interval $[0, 1]$ and with augmentation equal to 1.

Example (Tossing a fair coin)

Let G be an infinite cyclic group generated by x and consider

$$\mu := \tfrac{1}{2}x^{-1} + \tfrac{1}{2}x.$$

After n tosses, the probability that $\#Heads - \#Tails = m$ is equal to the coefficient of x^m in

$$\mu^n = \left(\tfrac{1}{2}x^{-1} + \tfrac{1}{2}x\right)^n = \frac{1}{2^n}\sum_{j=0}^{n}\binom{n}{j}x^{n-2j}.$$

In the coin toss, the number of head and tails can be equal if $n = 2m$ is even. The return probability is then given by

$$\frac{1}{2^{2m}}\binom{2m}{m}.$$

We can estimate this by using Stirling's formula $n! \sim \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}$.

$$\frac{1}{2^{2m}}\binom{2m}{m} \sim \frac{1}{\sqrt{\pi m}}.$$

Theorem (Kesten, 1959)

Let G be a finitely generated group with a finitely symmetric probability measure μ which support generates G. Then the return probability decays exponentially if and only if G is non-amenable.

So ... to find unusual behviour of return probabilities we might look at the class of amenable groups ... and soluble groups are amenable.

Theorem (Kropholler–Lorensen, 2019)

Let G be a finitely generated polyminimax group. Then there is a surjective group homomorphism $G^* \to G$ where G^* is torsion-free and polyminimax.

Theorem (Pittet–Saloff-Coste 2003, Kropholler–Lorensen 2019)

Let G be a polyminimax group that is generated by a finite set X. Assume that $1 \in X = X^{-1}$. Then

$$P_{G,S}(n) \succeq \exp(n^{1/3}).$$