Algebras defined by Lyndon words and Artin-Schelter regularity

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AS regular algebras were introduced and studied first in [AS, ATV1, ATV2] in 90's. The problems of classification and finding new classes of regular algebras are central for noncommutative algebraic geometry. When $d \leq 3$ all regular algebras are classified. The problem of classification is difficult and remains open even for regular algebras of $gl\dim = 5$.

Theorem

Let $A = \mathbf{k}\langle X \rangle / (\Re)$ *be a quantum binomial algebra,* $|X| = n$
The following conditions are equivalent:

(1) *A is an Artin-Schelter regular algebra, where* $\Re$ *is a Gr¨bner basis.*

(2) *A is a Yang-Baxter algebra, that is the automorphism* $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ *is a solution of the Yang-Baxter equation.*

(3) *A is a binomial skew polynomial ring, with respect to some enumeration of X.*

(3) *The Hilbert series of A is*

$$H_A(z) = \frac{1}{(1-z)^n}.$$

Each of these conditions implies that A is Koszul and a Noetherian domain.

Definition

Let $V = Span_{\mathbf{k}} X$. Let $\Re \subset \mathbf{k}\langle X \rangle$ be a set of quadratic binomials, satisfying the following conditions:

B1 Each $f \in \Re$ has the shape $f = xy - c_{yx} y'x'$, where $c_{xy} \in \mathbf{k}^{\times}$ and $x, y, x', y' \in X$.

B2 Each monomial xy of length 2 occurs at most once in $\Re$.

The (involutive) automorphism $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ *associated with* $\Re$ is defined as

$$R(x \otimes y) = c_{xy} y' \otimes x', \text{ and } R(y' \otimes x') = (c_{xy})^{-1} x \otimes y$$
$$\text{iff } xy - c_{xy} y'x' \in \Re.$$
$$R(x \otimes y) = x \otimes y \text{ iff } xy \text{ does not occur in } \Re.$$

The algebra $A = \mathbf{k}\langle X \rangle / (\Re)$ is *a quantum binomial algebra* if the relations are *square-free* and the associated quadratic set (X, r) is nondegerate. A is *a Yang-Baxter algebra* (Manin,1988), if the map $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$, is a solution of the YBE, $R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23}$.

Settings: $X = \{x_1, \cdots, x_n\}$ is a finite alphabet; K is a field, W is an antichain of monomials in X^+

We study classes $\mathfrak{C}(X, W)$ consisting of associative graded K-algebras $A = K\langle X\rangle / I$ generated by X and with a fixed obstructions set W. Our study is related to the following (at least):

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True, whenever the monomial algebra $A_W \in \mathcal{C}(X, W)$ has finite global dimension, see Theorem A.

Definitions and Conventions.

Let $X = \{x_1, x_2, \cdots, x_n\}$ be a finite alphabet. X^* and X^+ denote, resp., the free monoid, and the free semigroup generated by X, ($X^+ = X^* - \{1\}$). We consider two orderings on X^*.

1. The *lexicographic order* $<$ on X^+, $x_1 < x_2 < \cdots < x_n$.
   $u < v$ *iff* either $v = ub, b \in X^+$, or

$$u = axb, \; v = ayc \text{ with } x < y, \; x, y \in X, \; a, b, c \in X^*.$$

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Conv. All Gröbner bases of (associative) ideals I in $K\langle X\rangle$ and all Lyndon-Shyrshov Lie bases of Lie ideals J in $Lie(X)$ will be considered with respect to "$\prec$"-the deg-lex well-ordering on X^*.

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Def. A word $a \in X^*$ is *W-normal* (*W-standard*) if $u \not\subseteq a, \forall u \in W$.

$$\mathfrak{N} = \mathfrak{N}(W) := \{a \in X^* \mid a \text{ is } W\text{-normal}\}.$$

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Def. Let I be an ideal in $K\langle X\rangle$, $A = K\langle X\rangle/I$, $\overline{I}$ the set of all highest monomials of elements of I, w.r.t. $\prec$. *The set of obstructions* $W = W(I)$ is the subset of all words in $\overline{I}$ which are minimal w.r.t. $\sqsubseteq$:

$$W(I) = \{u \in \overline{I} \mid v \sqsubseteq u,\ \ v \in \overline{I} \Longrightarrow v = u\}.$$

W is the unique maximal antichain of monomials in $\overline{I}$.

Remarks

$W(I)$ depends on the ideal I, as well as, on the order $\prec$ on X^+. Let $A = K\langle X\rangle/I$. The theory of Gröbner bases implies that there is an isomorphism of K-vector spaces

$$K\langle X\rangle = Span_K \mathfrak{N}(W) \bigoplus I, \quad A \cong Span_K \mathfrak{N}(W).$$

$W = W(I)$ is also called *the set of obstructions for* $\mathfrak{N}$, or *the set of obstructions for* A.

NB. It is known that the ideal I has *unique reduced Groebner basis*

$G_0 = \{f_u = u + h_u \mid u \in W, \ \overline{h} \prec u \ h_u$ in normal form mod $G_0 - f_u\}$,

In other words, $W = \overline{G_0}$.

Example. $X = \{x < y\}$, $W = \{xxy,\ xyy\}$, is an antichain of Lyndon words,the class $\mathfrak{C}(X, W)$ contains two non-isomorphic regular algebras: A, B.

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I = (f_1, f_2), & f_1 = x^2y - yx^2, \quad \overline{f_1} = xxy \in W \\
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\mathfrak{R} = \{f_1, f_2\} & \text{is the reduced Gröbner basis of } I \\
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$A \in \mathfrak{C}(X, W)$ is an AS-regular algebra of $gl\dim A = 3$, **type A**.

The same class $\mathfrak{C}(X, W)$, with $X = \{x < y\}$, $W = \{xxy, \quad xyy\}$, $N = \{x < xy < y\}$

(ii) $B = K\langle x, y\rangle / I \in \mathfrak{C}(X, W)$,

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$B = U\mathfrak{g}$, the enveloping algebra of the 3-dimensional Lie algebra $\mathfrak{g} = Lie(x,y)/([xxy],[xyy])_{Lie}$, with a K-basis $[N] = \{x, [x,y], y\}$, hence B is AS regular.

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$\mathfrak{g} \simeq \mathfrak{h}_3$, the 3-dimensional *Heisenberg algebra* with a K-basis x, y, t, and relations $[x, y] = t$, $[x, t] = 0$, $[y, t] = 0$.

First results-the general case when $A = K\langle X\rangle/I$ in $\mathfrak{C}(X, W)$ has polynomial growth and finite global dimension. We prove that

Given the class $\mathfrak{C}(X, W)$, such that the monomial algebra $A_W = K\langle X\rangle/(W) \in \mathfrak{C}(X, W)$ has $gl \dim A_W = d < \infty$, $GK \dim A_W < \infty$. Then

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In particular, A is standard finitely presented (s.f.p.).
Remark. In general, $gl.\dim A \leq gl.\dim A_W$ (always) and I have examples when $gl.\dim A < gl.\dim A_W$. Surprisingly, when A_W has $gl\dim A_W = d < \infty$ and polynomial growth, the global dimension *$gl.\dim A$ does not depend on the shape of the defining relations* of A but only *on the set of obstructions* W.

Anick. The set of n-chains on W is defined recursively.

A (-1)-chain is the monomial 1, a 0-chain is any element of X, and a 1-chain is a word in W. An $(n+1)$-prechain is a word $w \in X^+$, which can be factored in two different ways $w = uvq = ust$ such that $t \in W$, u is an $n-1$ chain, uv is an n-chain, and s is a proper left segment of v. An $(n+1)$-prechain is an $(n+1)$-chain if no proper left segment of it is an n-prechain. In this case the monomial q is called *the tail of the n-chain* w.

Theorem [Anick] Suppose $W \subset X^+$ is an antichain of monomials. The monomial algebra $A_W = K\langle X\rangle/(W)$ has $gl \dim A_W = d$ *iff* there are no d-chains on W but there exists a $d-1$ chain on W.

Theorem I. Suppose W is an antichain in X^+,

s.t. there are no d-chains but there exists a $d-1$-chain on W .
Let $A = K\langle X\rangle / I \in \mathfrak{C}(X, W)$, with $GK \dim A < \infty$. Then:

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(2.a) The normal K-basis $\mathfrak{N}$ of A and its Hilbert series satisfy:
$\mathfrak{N} = \{a_1^{k_1} a_2^{k_2} \cdots a_d^{k_d} \mid k_i \geq 0,\ 1 \leq i \leq d\}$,
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(2.c) A is *s.f.p.*, the ideal I has a finite reduced Gröbner basis $\mathfrak{R}$, where $|\mathfrak{R}| = |W| \leq d(d-1)/2$.

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(a) $|W| = d(d-1)/2$; (b) $d = gl\dim A = n$; (c) A is a PBW algebra (Priddy); (d) $M = X$ and $\exists$ possibly new, ordering of X, $X = \{y_1 \leftarrow \cdots \leftarrow y_n\}$, s.t. $W = \{y_j y_i \mid 1 \leq i < j \leq n\}$.

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In this case A is Koszul.

Classes $\mathfrak{C}(X, W)$, where W is an anticahin of Lyndon words are of special interest.

A nonperiodic word $u \in X^+$ is *a Lyndon word* if it is minimal (with respect to $<$) in its conjugate class

$$u = ab, a, b \in X^+ \implies u < ba.$$

L denotes the set of Lyndon words in X^+. By definition $X \subset L$.

Example. $X = \{x < y\}$. The Lyndon words of length ≤ 5 are:

$$\begin{array}{l} x, y, xy, xxy, xyy, \\ xxxy, xxyy, xyyy, \\ xxyxy, xyxyyy, \quad x^k y^l, k + l = 5, 1 \leq k, l \leq 4. \end{array}$$

Some Facts. (1) If $a < b$ are Lyndon words then ab is a Lyndon word, so $a < ab < b$.

(2) Let $w \in L$. If $w = ab$, where b is the longest proper right segment of w with $b \in L$ then $a \in L$. This is *the (right) standard factorization of w* and denoted as $w = (a, b) = (a, b)_r$.(Used for the standard Lie bracketing of Lyndon words).

Obstructions set W, Lyndon Atoms $N = N(W)$. Duality $W \longleftrightarrow N(W)$

Definition. Given an antichain W of Lyndon words, the set of *W-normal Lyndon words* is denoted by $N = N(W)$, and is called a *the set of Lyndon atoms corresponding to W.*

$$N = N(W) = \mathfrak{N}(W) \bigcap L.$$

We study classes $\mathfrak{C}(X, W)$ of associative graded K-algebras A generated by X and with a fixed obstructions set W consisting of *Lyndon words in the alphabet* X. Clearly, the monomial algebra $A_{mon} = K\langle X\rangle/(W) \in \mathfrak{C}(X, W)$. Moreover, all algebras A in $\mathfrak{C}(X, W)$ share the same PBW type K-basis $\mathfrak{N}$, built out of the *Lyndon atoms* N. In general, the set N may be infinite. N "controls" the $GK \dim A$, and W "controls" $gl \dim A_{mon}$: A has polynomial growth of degree d *iff* $|N| = d$, moreover $gl \dim A \leq gl \dim A_W \leq |W| - 1$, whenever W is a finite set.

Relations between W and $N(W)$, Lyndon pairs (N, W)

Each antichain $W \subset L$ determines uniquely a set of Lyndon atoms $N = N(W) \subset L$. It satisfies

$$\begin{array}{ll} \textbf{C1.} & X \subseteq N. \\ \textbf{C2.} & \forall v \in L, \forall u \in N, v \sqsubseteq u \Longrightarrow v \in N. \\ \textbf{C3.} & u \in N \Longleftrightarrow u \in L \text{ and } u \notin (W). \end{array}$$

Conversely, each set N of Lyndon words satisfying conditions **C1** and **C2** determines uniquely an antichain of Lyndon monomials $W = W(N)$, such that condition **C3** holds, and N is exactly the set of Lyndon atoms corresponding to W.
In this case (N, W) will be called *a Lyndon pair*.

Open Question 1. Is it true that if A is an (s.f.p.) Artin-Schelter regular algebra there exists an appropriate ordering $<$ on X, so that the obstructions set W of A consists of Lyndon words?

True for the class of $\mathbb{Z}^2$-graded AS-regular algebras $A = K\langle x_1, x_2\rangle / I$ of global dimension 5.
(Floystad-Watne,2011, G.S. Zhou, D.M. Lu, 2013)

Open Question 2. Let (N, W) be a Lyndon pair in X^+. When the class $\mathfrak{C}(X, W)$ contains AS-regular algebras?

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(1) If "Yes" then N is finite, $|N| = d$, hence $|W| \leq d(d-1)/2$. This is not sufficient.

(2) $\mathfrak{C}(X, W)$ contains an abundance of AS-regular algebras, whenever $|W| = d(d-1)/2$. Here $N = X$, $gl \dim A = n$ (Thm II) e.g. $n = 8$, $\mathfrak{C}(X, W)$ contains ≥ 2400 AS-reg.alg.

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(4) $\mathfrak{C}(X, W)$ contains an AS-regular algebra of $gl \dim A = |N|$, whenever $\mathfrak{g} = Lie(X)/([W])$, has a K-basis $[N]$, or equivalently $[W]$ is a GS-Lie basis (this can be effectively verified). Here $A = U\mathfrak{g}$. In this case N is connected.

(5) $\mathfrak{C}(X, W(Fib_6))$ contains the monomial Fibonacci -Lyndon algebra F_6, $GK\dim F_6 = gl.\dim F_6 = 6$, but does not contain a $\mathbb{Z}_2$-graded AS-regular algebras.

Proposition 1.

- (1) There exists a one-to-one correspondence between the set $\mathbb{W}$ of all antichains W of Lyndon words with $X \bigcap W = \emptyset$ and the set $\mathbb{N}$ of all sets $N \subset L$ satisfying **C1** and **C2**.

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\phi : & \mathbb{W} \longrightarrow \mathbb{N} & & W \mapsto N(W) \\
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- (4) Each $N \in \mathbb{N}$ determines uniquely $A_{mon} = K\langle X\rangle/(W)$, with def. relations $W = W(N)$ and Lyndon atoms N.

$$GK\dim A_{mon} = d \Longleftrightarrow |N| = d.$$

Suppose $N = \{l_1 < l_2 < l_3 \cdots < l_d\}$ is a set of Lyndon words closed under taking Lyndon subwords.
$m = \max\{|l_i| \mid 1 \le i \le d\}$

We say that N is connected if

$$N \bigcap L_s \neq \emptyset, \quad \forall s \le m.$$

This is a necessary condition for "$\mathfrak{S}(X, W)$ contains the enveloping algebra $U = U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$".

Lemma. Suppose (N, W) is a Lyndon pair. If the class $\mathfrak{S}(X, W)$ contains the enveloping algebra $U = U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ then N is a connected set of Lyndon atoms, and $[N]$ is a K-basis for $\mathfrak{g}$.

Theorem II. Let $W \subset X^+$ be an antichain of monomials, let N be the set of normal Lyndon word, $N = \mathfrak{N} \bigcap L$ is not necessarily finite.

(1) W is an anticain of Lyndon words *iff* the set of normal words $\mathfrak{N} = \mathfrak{N}(W)$ has the shape

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Suppose $\mathfrak{g} = \mathrm{Lie}(X)/J$ is a Lie algebra, $U = U\mathfrak{g} = K\langle X\rangle/I$. Let W be the set of obstructions for U, let $A_W = K\langle X\rangle/(W)$, $\mathfrak{N} = \mathfrak{N}(I)$, $N = N(W) = \mathfrak{N} \bigcap L$.

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Proposition 2.

Suppose $\mathfrak{g} = \text{Lie}(X)/J$ is a Lie algebra, $U = U\mathfrak{g} = K\langle X\rangle/I$. Let W be the set of obstructions for U, let $A_W = K\langle X\rangle/(W)$, $\mathfrak{N} = \mathfrak{N}(I)$, $N = N(W) = \mathfrak{N} \bigcap L$.

(1) (N, W) is a Lyndon pair, N is connected. A_W is *a monomial algebra defined by Lyndon words*, in the sense of GIF.

(2) Assume that J is generated by homogeneous Lie elements, so U is canonically graded. Then U, and A_W are in the class $\mathfrak{C}(X, W)$. FAEQ.

  (a) *$U = U\mathfrak{g}$ is an Artin-Schelter regular algebra.*

  (b) The algebra U has polynomial growth.

  (c) The Lie algebra $\mathfrak{g}$ is finite dimensional.

  (d) The set of Lyndon atoms N is finite.

(3) Each of these equiv. conditions implies that U is s.f.p., and

$$d - 1 \leq |W| \leq d(d-1)/2, \quad \text{where } d = |N|,$$

$$gldim(U) = GK\dim(U) = \dim_K \mathfrak{g} = |N| = d.$$

Theorem III.

Let (N, W) be a Lyndon pair, $|N| = d$, so $d - 1 \leq |W| \leq \frac{d(d-1)}{2}$.

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$\mathfrak{C}(X, W)$ contains an abundance of (non isomorphic) PBW AS regular algebras: each of them is a skew polynomial ring with square-free binomial relations (GI), and defines a solution of the YBE.

Monomial Lie algebras

Let W be an antichain of Lyndon words, let $J = ([W])_{Lie}$ be the Lie ideal
generated by $[W] = \{[w] \mid w \in W\}$ in $\mathrm{Lie}(X)$. The Lie algebra $\mathfrak{g} = \mathrm{Lie}(X)/J$ is called *a monomial Lie algebra defined by Lyndon words*, or shortly, *a monomial Lie algebra*. We call $\mathfrak{g}$ a *a standard monomial Lie algebra* and denote it by $\mathfrak{g}_W$ if $[W]$ is a Gröbner-Shirshov basis of the Lie ideal J. In this case $U\mathfrak{g} \in \mathfrak{C}(X, W)$.

Theorem IV

$X = \{x < y\}$, (N_s, W_s) is a Lyndon pair in X^+. $J_s = ([W_s])_{Lie}$
$\mathfrak{g}_s = Lie(X)/J_s$, I_s is the two-sided ideal $I_s = ([W_s]_{ass})$ in $K\langle X\rangle$,
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Monomial Lie algebras of dimension 6

- $\mathfrak{g} \in \mathfrak{N}_4$

$$
\begin{array}{lll}
(6.4.1) & N & x < x^3y < x^2y < xy < xy^2 < y \\
 & W & x^4y,\ x^3yx^2y,\ x^2yxy,\ xyxy^2,\ xy^3 \\
(6.4.2) & N & x < x^2y < x^2y^2 < xy < xy^2 < y \\
 & W & x^3y,\ x^2yx^2y^2,\ x^2y^2xy,\ xyxy^2\ xy^3
\end{array}
$$

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$$\begin{array}{lll} (6.4.1) & N & x < x^3y < x^2y < xy < xy^2 < y \\ & W & x^4y,\ x^3yx^2y,\ x^2yxy,\ xyxy^2,\ xy^3 \\ (6.4.2) & N & \quad x < x^2y < x^2y^2 < xy < xy^2 < y \\ & W & x^3y,\ x^2yx^2y^2,\ x^2y^2xy,\ xyxy^2\ xy^3 \end{array}$$

- $\mathfrak{g} \in \mathfrak{N}_5$, Filiform Lie algebras of dimension 6.

$$\begin{array}{lll} (6.5.3) & N & x < xy < xy^2 < xy^3 < xy^4 < y \\ & W & xy^ixy^{i+1},\ 0 \leq i \leq 3,\ xy^5 \\ (6.5.4) & N & x < xy < xyxy^2 < xy^2 < xy^3 < y \\ & W & x^2y,\ xyxyxy^2,\ xyxy^2xy^2,\ xy^2xy^3,\ xy^4 \end{array}$$

Each of the remaining classes $\mathfrak{C}(X,W)$ *does not* contain enveloping of a Lie algebra.

(6.5.5)	N	$x < x^2y < x^2yxy < xy < xy^2 < y$
	W	$x^3y, x^2y^2, (x^2y)^2xy,\ x^2y(xy)^2, xyxy^2, xy^3$
(6.7.6)	N	$x < x^3y < x^3yx^2y < x^2y < xy < y$
	W	$x^4y,\ (x^3y)^2x^2y,\ x^3y(x^2y)^2, x^2yxy, xy^2$
(6.7.7)	N	$x < x^2y < x^2yxy < x^2yxyxy < xy < y$
	W	$x^3y, (x^2y)^2xy,\ (x^2yxy)^2xy,\ x^2y(xy)^3, xy^2$
(6.7.8)	N	$F_6\ x < xy < xyxy^2 < xyxy^2xy^2 < xy^2 < y$
	W	$x^2y,\ xyxyxy^2,\ xyxy^2xy(xy^2)^2,\ xy(xy^2)^3,\ xy^3$
Fibonacci algebra		

Standard Monomial Lie algebras of dimension 7;[W] is a GS basis only for N_1 through N_9

$$
\begin{aligned}
&(7.4.1) && N_1 = \{x < x^3y < x^2y < x^2y^2 < xy < xy^2 < y\},\ m = 4;\\
&(7.4.2) && N_2 = \{x < x^3y < x^2y < xy < xy^2 < xy^3 < y\},\ m = 4\\
&(7.4.3) && N_3 = \{x < x^2y < x^2y^2 < xy < xy^2 < xy^3 < y\},\ m = 4\\
&(7.5.4) && N_4 = \{x < xy < xyxy^2 < xy^2 < xy^3 < xy^4 < y\},\ m = 5\\
&(7.5.5) && N_5 = \{x < x^2y < xy < xy^2 < xy^3 < xy^4 < y\},\ m = 5\\
&(7.5.6) && N_6 = \{x < x^2y < xy < xyxyy < xy^2 < xy^3 < y\},\ m = 5\\
&(7.5.7) && N_7 = \{x < x^2y < x^2yxy < x^2y^2 < xy < xy^2 < y\},\ m = 5\\
&(7.5.8) && N_8 = \{x < x^2y < x^2y^2 < xy < xyxy^2 < xy^2 < y\},\ m = 5\\
&(7.6.9) && N_9 = \{x < xy < xy^2 < xy^3 < xy^4 < xy^5 < y\},\ m = 6\\
& && W_9 = \{xy^ixy^{i+1}, 0 \le i \le 4\} \bigcup \{xy^6\}\\
&(7.5.10)* && N = \{x < x^2y < x^2yxy < xy < xy^2 < xy^3 < y\}\\
&(7.5.11)* && N = \{x < x^3y < x^2y < xy < xyxy^2 < xy^2 < y\}\\
&(7.6.12)* && N = \{x < xy < xyxy^2 < xyxy^3 < xy^2 < xy^3 < y\}
\end{aligned}
$$

$$(7.5.13) \quad N = \{x < x^2y < x^2yxy < xy < xyxy^2 < xy^2 < y\}$$
$$(7.6.14) \quad N = \{x < x^2y < x^2y^2 < x^2y^2xy < xy < xy^2 < y\}$$
$$(7.7.15) \quad N = \{x < xy < xy^2 < xy^2xy^3 < xy^3 < xy^4 < y\}$$
$$(7.7.16) \quad N = \{x < xy < xyxy^2 < xy^2 < (xy^2)(xy^3) < xy^3 < y\}$$
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$$(7.7.18) \quad N = \{x < xy < (xy)(xyxy^2) < xyxy^2 < xy^2 < xy^3 < y\}$$
$$(7.7.19) \quad N = \{x < x^2y < (x^2y)(x^2y^2) < x^2y^2 < xy < xy^2 < y\}$$
$$(7.7.20) \quad N = \{x < x^2y < xy < xy(xyxy^2) < xyxy^2 < xy^2 < y\}$$
$$(7.8.21) \quad N = \{x < x^2y < xy < xyxy^2 < (xyxy^2)(xy^2) < xy^2 < y\}$$
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$$(7.9.25) \quad N = \{x < xy < xyxyxyxy^2 < xyxyxy^2 < xyxy^2 < xy^2 < y\}$$
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$$(7.11.27) \quad N = \{x < xy < xy^2 < xy^2xy^3 < (xy^2xy^3)(xy^3) < xy^3 < y\}$$
$$(7.11.28) \quad N = \{x, xy, xyxy^2, xyxy^2xy^2, xyxy^2xy^2xy^2, xy^2 < y\}$$
$$(7.12.29) \quad N = \{x, xy, xyxyxy^2, xyxyxy^2xyxy^2, xyxy^2, xy^2 < y\}$$
$$(7.13.30) \quad N_{F_7} = \{x, y, xy, xyy, xyxyy, xyxyyxyy, xyxyyxyxyyxyy\}.$$