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Groups with all Subgroups Permutable or Soluble of Finite Rank

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Introduction

The theory of groups all of whose subgroups satisfy some property has a very long history, beginning in 1897 with R. Dedekind

(1831 – 1916)

Richard Dedekind
"*Über Gruppen deren sämtliche Teiler Normalteiler sind*",
Math. Ann. **48** (1897), 548–561.

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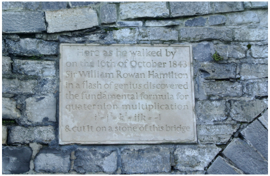
Obviously all abelian groups are Dedekind groups and a non-abelian Dedekind group is called Hamiltonian group.

Introduction

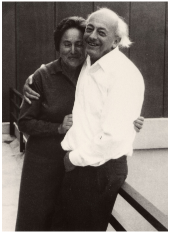
The smallest example of a Hamiltonian group is the quaternion group of order 8, denoted by Q_8 and discovered by William R. Hamilton in 1843.



"Here as he walked by on the 16th of October 1843 Sir William Rowan Hamilton in a flash of genius discovered the fundamental formula for quaternion multiplication $i^2 = j^2 = k^2 = ijk = -1$ *& cut it on a stone of this bridge"*

Introduction

However, every Hamiltonian group contains a copy of Q_8, in fact and a classical result of R. Baer and R. Dedekind (in the infinite and respectively finite order case)...



(1902 – 1979)

Reinhold Baer
"*Situation der Untergruppen und Struktur der Gruppe*",
S.B. Heidelberg. Akad. Wiss. **2** (1933), 12–17.

Introduction

. . . shows that every Hamiltonian group is a direct product of the form

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where

- B is an elementary abelian 2-group

Introduction

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$$G = Q_8 \times B \times D$$

and

- D is a periodic abelian group with all elements of odd order

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A subgroup H of a group G is said to be permutable in G if

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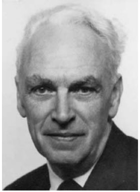
for every subgroup K of G.

These subgroups had been introduced by Øystein Ore in 1937, who called them "quasinormal".



(1899 – 1968)

Øystein Ore
"*Structures and group theory*",
I. Duke Math. J.**3** (1937), 149–173.

Obviously, normal subgroups are always permutable, but the converse is not true.

Obviously, normal subgroups are always permutable, but there are numerous example of non-normal permutable subgroups.

Example

Let p be an odd prime, and let G be an extraspecial group of order p^3 and exponent p^2.

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Example
Let p be an odd prime, and let G be an extraspecial group of order p^3 and exponent p^2.

- G has all subgroups permutable
- G has non-normal subgroups

Introduction

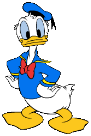
Groups with all subgroups permutable are called quasi-Hamiltonian (or quasi-Abelian) groups and they were studied by Kenkichi Iwasawa who classified them.

(1917– 1998)

Kenkichi Iwasawa
"*Einege sätze über freie gruppen*",
Proc. Imp. Acad.Tokyo **19** (1943),
272–274.

Introduction

What about the structure of groups in which the set of all non-permutable subgroups is "small" in some sense?



Introduction

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I examined this problem together with **Martyn R. Dixon**, **Zekeriya Yalcin Karatas** and **Marco Trombetti** in the wake of the following paper

S. Franciosi, F. de Giovanni and M. L. Newell
"*Groups with Polycyclic Non-Normal Subgroups*",
Algebra Colloq. **7** (2000), no. 1, 33-42.

Dually to their results for the non-normal subgroups, we have proved that

- If G is a group whose non-permutable subgroups are periodic, then G is quasihamiltonian or periodic

Dually to their results for the non-normal subgroups, we have proved that

- If G is a group whose non-permutable subgroups are periodic, then G is quasihamiltonian or periodic
- If G is locally graded group and the non-permutable subgroups of G are locally finite, then G is quasihamiltonian or locally finite

Preliminaries

Dually to their results for the non-normal subgroups, we have proved that

- If G is a group whose non-permutable subgroups are periodic, then G is quasihamiltonian or periodic
- If G is locally graded group and the non-permutable subgroups of G are locally finite, then G is quasihamiltonian or locally finite
- If G a locally graded group whose subgroups are permutable or Černikov, then G is quasihamiltonian or Černikov

Preliminaries

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Why a locally graded group?

Preliminaries

- If G is locally graded group and the non-permutable subgroups of G are locally finite, then G is quasihamiltonian or locally finite

- If G a locally graded group whose subgroups are permutable or Černikov, then G is quasihamiltonian or Černikov

Why a locally graded group?

A group G is said to be locally graded if all its finitely generated nontrivial subgroups have a nontrivial finite image and the consideration of Tarski groups shows that a requirement of this type is necessary for this kind of problem

In 1967, S.N. Černikov showed that an infinite locally graded group whose infinite subgroups are normal

(1912 – 1987)

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"*Groups with given properties of system of infinite subgroups*",
Ukrain. Math. J. **19** (1967), 715-731.

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In 1967, S.N. Černikov showed that an infinite locally graded group whose infinite subgroups are normal is either a Dedekind group or an extension of a Prüfer group by a finite Dedekind group

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Generalizing this result, we have proved that in an infinite locally graded group G,

every infinite subgroup is permutable if and only if G is quasihamiltonian

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Ok! But, it didn't occur to you it might exist further generalizations?

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Of course!

Finite abelian section rank

The first more general property we consider is the finiteness for abelian section rank.

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A group G has finite abelian section rank if every abelian section of G has finite 0-rank and finite p-rank for all primes p.

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1. G is quasihamiltonian;
2. G is soluble of finite abelian section rank;
3. G has finite abelian section rank and G'' is a finite perfect minimal non-soluble group such that G/G'' is quasihamiltonian.

Finite rank

A group G is said to have finite (Prüfer) rank $r = r(G)$ if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property.

Finite rank

As a corollary, we have that in a locally graded group G, all subgroups are permutable or soluble of finite rank if and only if either

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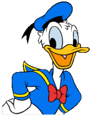
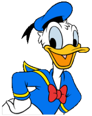
$\mathfrak{S}_1$-group



Do you also know what happens in a $\mathfrak{S}_1$-group?

$\mathfrak{S}_1$-group



Do you also know what happens in a $\mathfrak{S}_1$-group?

A group G is called an $\mathfrak{S}_1$-group if G is a hyperabelian group with finite abelian section rank and G contains elements of only finitely many distinct prime orders.

Thus, in particular, $\mathfrak{S}_1$-groups are certain types of soluble group with finite special rank.

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Let G be a locally graded group. Then all subgroups are permutable or $\mathfrak{S}_1$ if and only if either

1. G is quasihamiltonian;
2. G is an $\mathfrak{S}_1$-group;
3. G is a finite extension of an $\mathfrak{S}_1$-group and G'' is a finite perfect minimal non-soluble group such that G/G'' is quasihamiltonian.

Minimax groups

Next we recall that a group G is called minimax if it has a finite subnormal series whose factors either satisfy min or max.

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Clearly every soluble minimax group is an $\mathfrak{S}_1$-group.

Minimax groups

Let G be a locally graded group. Then all subgroups are permutable or soluble minimax if and only if either

1. G is quasihamiltonian;
2. G is a soluble minimax group;
3. G is a minimax group and $G^{''}$ is a finite perfect minimal non-soluble group such that $G/G^{''}$ is quasihamiltonian.

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Minimax groups

Let G a soluble minimax group all of whose non-permutable subgroups are polycyclic.

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If the finite residual R of G is a direct product of at least two Prüfer subgroups, then G is quasihamiltonian.

Periodic case

Periodic groups of the type we are interested in are now easy to describe.

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Let G be an infinite non-quasihamiltonian periodic soluble minimax group and let J be the finite residual of G.

Then all non-permutable subgroups of G are polycyclic if and only if J is a Prüfer p-group for some prime p and G/J is a finite quasihamiltonian group.

Non-periodic case

Let G be a non-periodic soluble minimax group with nontrivial finite residual J and suppose that G is not quasihamiltonian.

If all non-permutable subgroups of G are polycyclic, then the following conditions hold:

Non-periodic case

Let G be a non-periodic soluble minimax group with nontrivial finite residual J and suppose that G is not quasihamiltonian.
If all non-permutable subgroups of G are polycyclic, then the following conditions hold:

1. J is a Prüfer p-group for some prime p
2. G/J is quasihamiltonian and the set T of elements of finite order in G is a subgroup such that G/T is torsion-free abelian
3. T/J is a finite abelian group

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   Furthermore either

4. Every abelian subgroup is min-by-max, or
5. G contains an infinitely generated torsion-free abelian subgroup.

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Let G be a non-periodic, residually finite soluble minimax group that is neither polycyclic nor quasihamiltonian.

If all non-permutable subgroups of G are polycyclic, then G is nilpotent and central-by-finite and splits over the torsion part.

Thank you for your attention!