A generalisation of pronormal subgroups of finite groups to Sylow permutability

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Contents

Introduction

The results of this talk have been obtained in collaboration with P. Longobardi and M. Maj (*J. Algebra Appl.*, 2020).

Introduction

All our groups will be finite.

Introduction

Permutability and Sylow permutabilty

Definitions

- If $H, K \leqslant G$, H permutes with K when $HK = KH$, that is, HK is a subgroup of G.
- H is permutable in G if H permutes with all subgroups of G.
- H is S-permutable in G if H permutes with all Sylow subgroups of G.

Introduction

T-groups, PT-groups, and PST-groups

$$\begin{array}{ccccc} \text{normality} & \Longrightarrow & \text{permutability} & \Longrightarrow & \text{S-permutability} \\ \Downarrow \ \Big\Vert \ \text{T-group} & & \text{Ore} \Downarrow \ \Big\Vert \ \text{PT-group} & & \text{Kegel} \Downarrow \ \Big\Vert \ \text{PST-group} \\ \text{subnormality} & & \text{subnormality} & & \text{subnormality} \end{array}$$

None of these implications is an equivalence.

$$(\text{T-groups}) \quad \subset \quad (\text{PT-groups}) \quad \subset \quad (\text{PST-groups})$$

Characterisations based on embedding properties

Pronormality, weak normality, and the subnormaliser condition

Definitions

1. H is pronormal in G if for every $g \in G$, H and H^g are conjugate in $\langle H, H^g \rangle$ (Hall, Cambridge lectures).
2. H is weakly normal in G if the condition $H^g \leqslant \mathrm{N}_G(H)$ implies that $g \in \mathrm{N}_G(H)$ (Müller, 1966).
3. H satisfies the subnormaliser condition in G (or is transitively normal or pseudonormal in G) if $H \trianglelefteq K \trianglelefteq L$ implies that $H \trianglelefteq L$ (Peng, 1971).

H pronormal in G $\Longrightarrow$ H weakly normal in G $\Longrightarrow$ H satisfies the subnormaliser condition in G

Characterisations based on embedding properties

Pronormality, weak normality, and the subnormaliser condition

Theorem (Peng, 1969, see also Kaplan, 2011)

Let G be a group. The following statements are equivalent:

1. *G is a soluble T-group.*
2. *For every prime number p, all p-subgroups of G are pronormal.*
3. *All subgroups of G are pronormal.*

Characterisations based on embedding properties

Pronormality, weak normality, and the subnormaliser condition

Theorem (Peng, 1969, see also Kaplan, 2011)

Let G be a group. The following statements are equivalent:

1. *G is a soluble T-group.*
2. *For every prime number p, all p-subgroups of G are pronormal.*
3. *All subgroups of G are pronormal.*

In this talk, we will abbreviate this in the following form:

Theorem

Pronormality characterises soluble T-groups.

Characterisations based on embedding properties

Pronormality, weak normality, and the subnormaliser condition

Theorem (see Ballester-Bolinches, Esteban-Romero, *J. Austral. Math. Soc.*, 2003)

Let p be a prime and let H be a p-subgroup of G. The following statements are equivalent.

1. *H is a pronormal subgroup of G.*
2. *H is a weakly normal subgroup of G.*
3. *H satisfies the subnormaliser condition in G.*
4. *H is normal in $\mathrm{N}_G(X)$ for every p-subgroup X such that $H \leqslant X$.*
5. *H is normal in $\mathrm{N}_G(S)$ for every Sylow p-subgroup S of G such that $H \leqslant S$.*

Characterisations based on embedding properties

Pronormality, weak normality, and the subnormaliser condition

Theorem (Ballester-Bolinches, Esteban-Romero, *J. Austral. Math. Soc.*, 2003)

1. *Weak normality characterises soluble T-groups.*
2. *The subnormaliser condition characterises soluble T-groups.*

Characterisations based on embedding properties

Weak (S-)permutability and the (S-)subpermutiser condition

Question

Can we extend pronormality, weak normality and the subnormaliser condition for permutability and Sylow permutability in such a way the corresponding properties characterise soluble PT-groups and soluble PST-groups?

Characterisations based on embedding properties

Weak (S-)permutability and the (S-)subpermutiser condition

Rewrite

$$H^g \leqslant \mathrm{N}_G(H) \iff H \trianglelefteq \langle H, H^g \rangle$$

$$g \in \mathrm{N}_G(H) \iff H \trianglelefteq \langle H, g \rangle$$

and change in the right hand side of the equivalences normality by permutability and S-permutability.

Characterisations based on embedding properties

Weak (S-)permutability and the (S-)subpermutiser condition

Definitions (Ballester-Bolinches, Esteban-Romero, *J. Austral. Math. Soc.*, 2003)

Let $H \leqslant G$.
H is weakly permutable (weakly S-permutable) in G when the following condition holds:
If $g \in G$ and H is permutable (S-permutable) in $\langle H, H^g \rangle$, then H is permutable (S-permutable) in $\langle H, g \rangle$.

Characterisations based on embedding properties

Weak (S-)permutability and the (S-)subpermutiser condition

Definitions (Ballester-Bolinches, Esteban-Romero, *J. Austral. Math. Soc.*, 2003)

Let $H \leqslant G$.

1. H satisfies the subpermutiser condition if whenever H is permutable in K and K is permutable in L, we have that H is permutable in L.
2. H satisfies the S-subpermutiser condition if whenever H is S-permutable in K and K is S-permutable in L, we have that H is S-permutable in L.

We have that H satisfies the (S-)subpermutiser condition if whenever H is subnormal in L, we obtain that H is (S)-permutable in L.

Characterisations based on embedding properties

Weak (S-)permutability and the (S-)subpermutiser condition

$$\begin{array}{ccc}
\text{weakly normal} & \Longrightarrow & \text{subnormaliser condition} \\
\Downarrow & & \Downarrow \\
\text{weakly permutable} & \Longrightarrow & \text{subpermutiser condition} \\
\Downarrow & & \Downarrow \\
\text{weakly S-permutable} & \Longrightarrow & \text{S-subpermutiser condition}
\end{array}$$

Characterisations based on embedding properties

Weak (S-)permutability and the (S-)subpermutiser condition

Theorem (Ballester-Bolinches, Esteban-Romero, *Acta Math. Hungar.*, 2003)

1. *Weak (S-)permutability characterises soluble PT-groups (PST-groups).*
2. *The (S-)subpermutiser condition characterises soluble PT-groups (PST-groups).*

Characterisations based on embedding properties

Extensions of pronormality

pronormal	$\Longrightarrow$	weakly normal	$\Longrightarrow$	subnormaliser condition	$\mathfrak{S} \cap \mathrm{T}$
		$\Downarrow$		$\Downarrow$	
		weakly permutable	$\Longrightarrow$	subpermutiser condition	$\mathfrak{S} \cap \mathrm{PT}$
		$\Downarrow$		$\Downarrow$	
		weakly S-permutable	$\Longrightarrow$	S-subpermutiser condition	$\mathfrak{S} \cap \mathrm{PST}$

Characterisations based on embedding properties

Extensions of pronormality

pronormal	$\Longrightarrow$	weakly normal	$\Longrightarrow$	subnormaliser condition		$\mathfrak{S} \cap \mathrm{T}$
		$\Downarrow$		$\Downarrow$		
propermutable?		weakly permutable	$\Longrightarrow$	subpermutiser condition		$\mathfrak{S} \cap \mathrm{PT}$
		$\Downarrow$		$\Downarrow$		
pro-S-permutable?		weakly S-permutable	$\Longrightarrow$	S-subpermutiser condition		$\mathfrak{S} \cap \mathrm{PST}$

Characterisations based on embedding properties

Extensions of pronormality

Theorem

H is pronormal in G if and only if for each $g \in G$ there exists $x \in \langle H, H^g \rangle^{\mathfrak{N}}$ such that $H^g = H^x$.

Idea: If $T = \langle H, H^g \rangle$, then $HT^{\mathfrak{N}}/T^{\mathfrak{N}}$ is pronormal and subnormal in $T/T^{\mathfrak{N}} \in \mathfrak{N}$, so $HT^{\mathfrak{N}} \trianglelefteq T$ and $T = HT^{\mathfrak{N}}$.

$$H^g = H^x \iff H^{gx^{-1}} = H \iff gx^{-1} \in \mathrm{N}_G(H) \iff H \trianglelefteq \langle H, gx^{-1} \rangle$$

Theorem

H is pronormal in G if and only if for each $g \in G$ there exists $x \in \langle H, H^g \rangle^{\mathfrak{N}}$ such that $H \trianglelefteq \langle H, gx^{-1} \rangle$.

Characterisations based on embedding properties

Extensions of pronormality

Definition

H is propermutable in G if and only if for each $g \in G$ there exists $x \in \langle H, H^g \rangle^{\mathfrak{N}}$ such that H is permutable in $\langle H, gx^{-1} \rangle$.

Definition

H is pro-S-permutable in G if and only if for each $g \in G$ there exists $x \in \langle H, H^g \rangle^{\mathfrak{N}}$ such that H is S-permutable in $\langle H, gx^{-1} \rangle$.

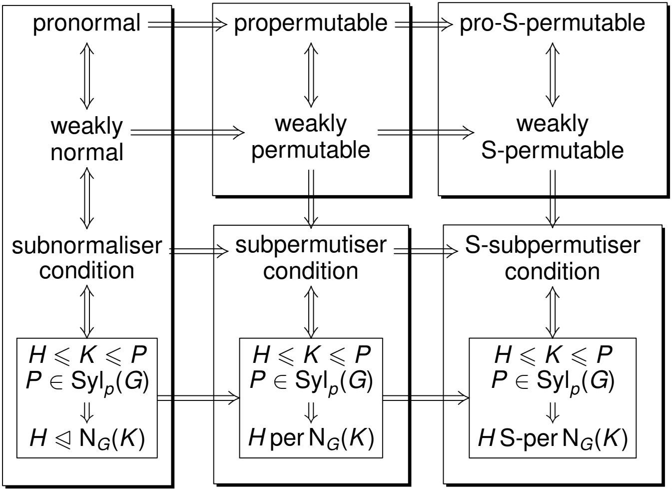
Characterisations based on embedding properties

Extensions of pronormality

					charact.
pronormal	$\Longrightarrow$	weakly normal	$\Longrightarrow$	subnormaliser condition	$\mathfrak{S} \cap \mathrm{T}$
$\Downarrow$		$\Downarrow$		$\Downarrow$	
propermutable	$\Longrightarrow$	weakly permutable	$\Longrightarrow$	subpermutiser condition	$\mathfrak{S} \cap \mathrm{PT}$
$\Downarrow$		$\Downarrow$		$\Downarrow$	
pro-S-permutable	$\Rightarrow$	weakly S-permutable	$\Rightarrow$	S-subpermutiser condition	$\mathfrak{S} \cap \mathrm{PST}$

Characterisations based on embedding properties

Extensions of pronormality: Equivalences for p-subgroups



Characterisations based on embedding properties

Extensions of pronormality: A final remark

We might have considered for propermutability and pro-S-permutability the following definitions:

H is α-propermutable (α-pro-S-permutable) in G if for each $g \in G$ there exists $x \in \langle H, H^g \rangle$ such that H is permutable (S-permutable) in $\langle H, gx^{-1} \rangle$.

- We have that α-propermutable subgroups also characterise soluble PST-groups.
- However, we have not been able to show that α-pro-S-permutable subgroups are weakly S-permutable.