# Groups whose non-Normal Subgroups are Metahamiltonian

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We will use $\mathfrak{H}$ to denote the class of metahamiltonian groups, which is trivially $SH$-closed, but not closed by extensions or even direct products.

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**Proposition**

Let $G$ be a group whose finitely generated subgroups are metahamiltonian. Then $G$ itself is metahamiltonian.

# Generalized metahamiltonian groups

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**Theorem 2.** [M. De Falco - F. de Giovanni - C. Musella, **2009**]

Let $G$ be a locally graded minimal-non-$\mathfrak{H}$ group. Then $G$ is finite.

Of course such a group is either perfect (one example is $A_5$) or soluble of derived length at most 4.

# Generalized metahamiltonian groups

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**Theorem 3. [M. De Falco - F. de Giovanni - C. Musella, 2009]**

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**Theorem 4. [M. De Falco, F. de Giovanni, C. Musella, 2009]**

Let $G$ be a locally graded group satisfying the minimal condition on non-metahamiltonian subgroups. Then $G$ is either Černikov or metahamiltonian.

# $k$-metahamiltonian groups

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We define, recursively, the class $\mathfrak{H}_k$ in the following way.
A group $G$ is in the class $\mathfrak{H}_k$ if and only if

$$\forall H \leq G, \ H \notin \mathfrak{H}_{k-1} \Rightarrow H \triangleleft G$$

and refer to groups in the class $\mathfrak{H}_k$ as *$k$-metahamiltonian groups*.
With $k = 1$ we have the usual metahamiltonian groups.
Obviously we have:

$$\mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_2 \subseteq \dots$$

# Properties of $k$-metahamiltonian groups

**Theorem 5.** [F. de Giovanni, D.E., M. Trombetti **2019**]

Let $G$ be a locally graded $\mathfrak{H}_k$-group. Then the commutator subgroup $G'$ is finite and if $G$ is soluble, its derived length does not exceed $3k$.

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Many other interesting properties of metahamiltonian groups can be proved also for $\mathfrak{H}_k$-groups:

- There exist non soluble $\mathfrak{H}_2$-groups, e.g. $A_5$, and $\mathfrak{H}_2$-groups whose commutator subgroup is not of prime-power order, e.g. $GL(2,3)$ whose commutator subgroup is $SL(2,3)$ which is a group of order $24 = 2^3 \cdot 3$.

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- Every locally graded minimal-non-$\mathfrak{H}_k$ group is finite.

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- Finitely generated $\mathfrak{H}_k$-groups are polycyclic-by-finite.
- Locally graded groups satisfying the minimal condition on non-$k$-metahamiltonian subgroups are necessarily Černikov or $k$-metahamiltonian.
- Polycyclic-by-finite groups whose finite homomorphic images all lie in the class $\mathfrak{H}_k$ are $\mathfrak{H}_k$.

# Groups with finitely many normalizers of non-$k$-metahamiltonian subgroups

F. De Mari and F. de Giovanni proved the following result concerning groups whose non-abelian subgroups have finitely many normalizers.

**Theorem 6. [F. de Giovanni - F. De Mari, 2006]**

Let $G$ be a group with finitely many normalizers of non-abelian subgroups. Then $G$ has finite commutator subgroup.

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This can be extended to

**Theorem 7. [F. de Giovanni, D.E., M. Trombetti 2019]**

Let $G$ be a group with finitely many normalizers of subgroups which are not $\mathfrak{H}_k$. Then $G$ has finite commutator subgroup.

# $\infty$-metahamiltonian groups

One can also define the class

$$\mathfrak{H}_\infty = \bigcup_{k \in \mathbb{N}} \mathfrak{H}_k.$$

Notice now that if $G$ is in this class, $G$ cannot have any infinite descending chain of subgroups:

$$G = G_0 > G_1 > G_2 > \dots \qquad \text{with } G_{i+1} \ntriangleleft G_i$$

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As a matter of fact, one can show that $\mathfrak{H}_\infty$ is precisely the class of groups with a bound on the length of such "bad" chains.
Notice that we already know that

$$\mathfrak{H}_\infty \subseteq \mathfrak{F}\mathfrak{A}$$

i.e. $\infty$-metahamiltonian groups have finite commutator subgroup.

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For if $G$ is a group such that $|G'| = n = p_1^{n_1} \ldots p_l^{n_l}$, then any descending chain $\{G_i\}$ of subgroups of $G$ related by $G_{i+1} \ntrianglelefteq G_i$ is bounded in length by $f(n) = n_1 + n_2 + \ldots + n_l$ and so $G$ is $\mathfrak{H}_{f(n)}$.

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The fact that a group with bound $f(n)$ on the length of this type of chains is $\mathfrak{H}_{f(n)}$ can be proved by induction on $n_1 + \dots + n_l$. The basis of induction is the observation that groups with commutator of prime order are necessarily metahamiltonian.

# Groups whose non-normal subgroups belong to a class $\mathfrak{X}$

Actually, we can start from any *SH*-closed class of groups $\mathfrak{X}$ and define:

$$\mathfrak{X}_0 = \mathfrak{X}$$

and, for any $k \in \mathbb{N}$, say that a group $G$ is in the class $\mathfrak{X}_k$ if and only if:

$$\forall H \leq G,\ H \notin \mathfrak{X}_{k-1} \Rightarrow H \triangleleft G.$$

We can then also define the class $\mathfrak{X}_\infty$ as $\bigcup_{k \in \mathbb{N}} \mathfrak{X}_k$.

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Then all properties in the list also hold for $\mathfrak{X}_k$ for any $k \in \mathbb{N}$

Thank you for the attention!!!