SKEW LATTICES AND SET-THEORETIC SOLUTIONS OF THE YANG-BAXTER EQUATION

(joint work with Karin Cvetko-Vah)

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SET-THEORETIC SOLUTIONS

Definition

*A **set-theoretic solution** of the Yang-Baxter equation is a pair (X, r) such that X is a non-empty set and $r : X \times X \to X \times X : (x, y) \mapsto (\sigma_x(y), \gamma_y(x))$ is a map where*

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- **Involutive**: $r^2 = id_{X^2}$
- **Idempotent**: $r^2 = r$
- **Cubic**: $r^3 = r$

SKEW LATTICES

Definition

*A **skew lattice** (SL) is a set S endowed with a pair of idempotent and associative operations $\wedge$ and $\vee$ which satisfy the absorption laws*

$$x \wedge (x \vee y) = x = x \vee (x \wedge y) \text{ and } (x \wedge y) \vee y = y = (x \vee y) \wedge y.$$

Notation: $(S, \wedge, \vee)$

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Examples

- Lattices

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Examples

- Lattices
- $(\{0, 1, 2\}, \wedge, \vee)$, where

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	2	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

STRONG DISTRIBUTIVE SOLUTIONS

Definition

A skew lattice $(S, \wedge, \vee)$ is called a ***strong distributive solution*** *of the Yang-Baxter equation if (S, r) is a set-theoretic solution of the Yang-Baxter equation, where*

$$r : S \times S \to S \times S : (x, y) \mapsto (x \wedge y, x \vee y).$$

Remark: (S, r) is cubic

STRONG DISTRIBUTIVE SOLUTIONS

{Strongly and co-strongly distributive SL}

$\rotatebox{-90}{$\subsetneq$}$

{Strong distributive solution}

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{Distributive and cancellative SL}

STRONG DISTRIBUTIVE SOLUTIONS

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$$\subsetneq$$

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$$\{\text{Distributive and cancellative SL}\}$$

Strongly distributive: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

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Distributive: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$
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Cancellative: $x \vee y = x \vee z$ and $x \wedge y = x \wedge z \Rightarrow y = z$
$x \vee z = y \vee z$ and $x \wedge z = y \wedge z \Rightarrow x = y$

LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

Definition

A skew lattice $(S, \wedge, \vee)$ *is called a* ***left (resp. right) distributive solution*** *of the Yang-Baxter equation if* (S, r) *is a set-theoretic solution of the Yang-Baxter equation, where*

$$r : S \times S \to S \times S : (x, y) \mapsto (x \wedge y, y \vee x) \quad (resp.\ (y \wedge x, x \vee y)).$$

Remark: (S, r) is idempotent

LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

$$\{\text{Left cancellative and distributive SL}\} = \{\text{Left distributive solution}\}$$

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LEFT AND RIGHT DISTRIBUTIVE SOLUTIONS

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$$=$$
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Left cancellative: $x \vee y = x \vee z$ and $x \wedge y = x \wedge z \Rightarrow y = z$

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Left cancellative: $x \vee y = x \vee z$ and $x \wedge y = x \wedge z \Rightarrow y = z$
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Left cancellative: $x \vee y = x \vee z$ and $x \wedge y = x \wedge z \Rightarrow y = z$
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Distributive: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$
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WEAK DISTRIBUTIVE SOLUTIONS

Definition

A skew lattice $(S, \wedge, \vee)$ *is called a* ***weak distributive solution*** *of the Yang-Baxter equation if* (S, r) *is a set-theoretic solution of the Yang-Baxter equation, where*

$$r : S \times S \to S \times S : (x, y) \mapsto (x \wedge y \wedge x, x \vee y \vee x).$$

Remark: (S, r) is idempotent

WEAK DISTRIBUTIVE SOLUTIONS

$$\{\text{Simply cancellative, distributive and lower symmetric SL}\}$$

$$=$$

$$\{\text{Weak distributive solution}\}$$

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Simply cancellative: $x \vee y \vee x = x \vee z \vee x$ and
$x \wedge y \wedge x = x \wedge z \wedge x \Rightarrow y = z$

WEAK DISTRIBUTIVE SOLUTIONS

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$$\{\text{Weak distributive solution}\}$$

Simply cancellative: $x \vee y \vee x = x \vee z \vee x$ and $x \wedge y \wedge x = x \wedge z \wedge x \Rightarrow y = z$

Distributive: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$

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WEAK DISTRIBUTIVE SOLUTIONS

$$\{\text{Simply cancellative, distributive and lower symmetric SL}\}$$
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$$\{\text{Weak distributive solution}\}$$

Simply cancellative: $x \vee y \vee x = x \vee z \vee x$ and $x \wedge y \wedge x = x \wedge z \wedge x \Rightarrow y = z$

Distributive: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$
$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$

Lower symmetric: $x \vee y = y \vee x \Rightarrow x \wedge y = y \wedge x$

SOLUTIONS FROM GENERAL SKEW LATTICES

Proposition

Let $(S, \wedge, \vee)$ be a skew lattice. Then, (S, r) is an idempotent set-theoretic solution of the Yang-Baxter equation, where

$$r : S \times S \to S \times S : (x, y) \mapsto ((x \wedge y) \vee x, y).$$

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Proposition

Let $(S, \wedge, \vee)$ be a skew lattice. Then, (S, r) is an idempotent set-theoretic solution of the Yang-Baxter equation, where

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Corollary

Let $(S, \wedge, \vee)$ be a skew lattice. The map $r(x, y) = (x\lfloor y \rfloor, y)$ is an idempotent set-theoretic solution of the Yang-Baxter equation.

$x\lfloor y \rfloor := (y \wedge x \wedge y) \vee x \vee (y \wedge x \wedge y)$ lower update of x by y

OVERVIEW

Strongly and co-strongly distributive SL

↓

Strong distributive solution

↓

Cancellative, distributive SL

OVERVIEW

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↓

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Cancellative, distributive SL

↙

Left cancellative, distributive SL
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Simply cancellative, distributive, lower symmetric SL
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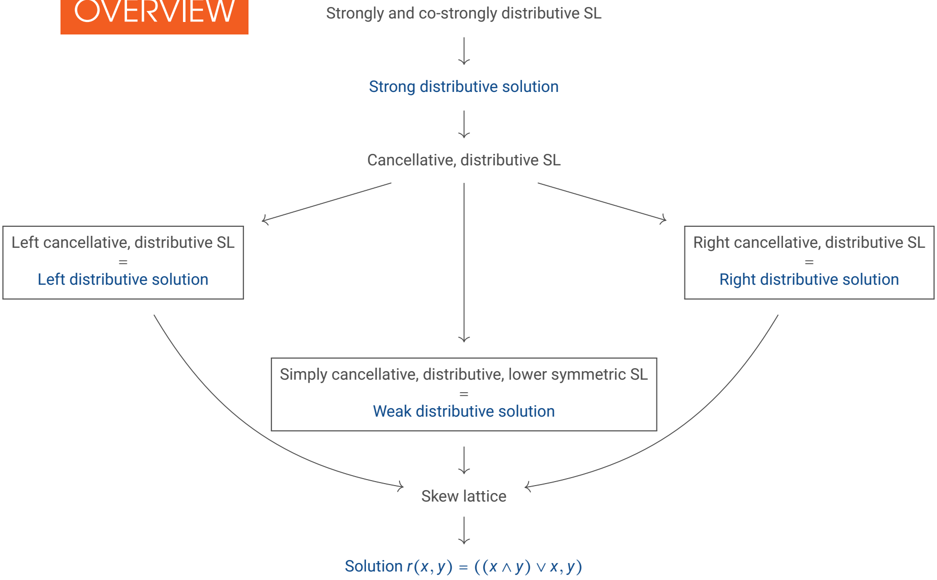
OVERVIEW


Strongly and co-strongly distributive SL
Strong distributive solution
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Weak distributive solution
Skew lattice
Solution $r(x,y) = ((x \wedge y) \vee x, y)$

QUESTION

Can we use skew lattices to generalize the notions of braces and cycle sets?

CONSTRUCTION

Proposition

Let $(I, \leq)$ be a totally ordered set, $\{A_i \mid i \in I\}$ a family of pairwise disjoint sets and $S = \dot\bigcup_{i\in I} A_i$. For any $i, j \in I$, $x \in A_i$ and $y \in A_j$ define

$$x \wedge y = \begin{cases} x & \text{if } i < j \\ y & \text{if } j \leq i \end{cases}, \qquad x \vee y = \begin{cases} y & \text{if } i < j \\ x & \text{if } j \leq i \end{cases}.$$

Then, $(S, \wedge, \vee)$ is a cancellative and distributive skew lattice.

Thank you for your attention!