On involutive square-free set-theoretic solutions of the Yang-Baxter equation

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Definition

A *set-theoretic solution of the Yang-Baxter equation* on a set X is a pair (X, r), where the map $r : X \times X \to X \times X$ is such that

$$r_1 r_2 r_1 = r_2 r_1 r_2,$$

where $r_1 := r \times id_X$ and $r_2 := id_X \times r$.

Problem (Drinfield, 1992)

Finding all set-theoretic solutions of the Yang-Baxter equation.

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Set-theoretic solutions of the Yang-Baxter equation

Convention: if X is a set and $r : X \times X \to X \times X$, we will denote by $\lambda_x(y)$ (resp. $\rho_x(y)$) the projection on the first component (resp. on the second component) of $r(x, y)$ (resp. of $r(y, x)$).

Definition

A set-theoretic solution of the Yang-Baxter equation $r : X \times X \to X \times X, \quad (x, y) \to (\lambda_x(y), \rho_y(x))$ is called:

1) *involutive* if $r^2 = id_{X\times X}$;
2) *non-degenerate* if $\lambda_x, \rho_x \in Sym(X)$ for every $x \in X$;
3) *square-free* if $r(x, x) = (x, x)$ for every $x \in X$.

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Basic definitions and results

Retractable square-free left cycle sets

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Set-theoretic solutions of the Yang-Baxter equation

Example

Let X be a non-empty set and $r : X \times X \to X \times X$ the function given by $r(x, y) := (y, x)$ for all $x, y \in X$. Then the pair (X, r) is an involutive non-degenerate square-free set-theoretic solution.

Convention: From now on, by a solution we mean a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.

Example

Let X be a non-empty set and $r : X \times X \to X \times X$ the function given by $r(x, y) := (y, x)$ for all $x, y \in X$. Then the pair (X, r) is an involutive non-degenerate square-free set-theoretic solution.

Convention: From now on, by a solution we mean a non-degenerate involutive set-theoretic solution of the Yang-Baxter equation.

Left cycle sets

Definition (Rump, 2005)

A pair $(X,\cdot)$ is said a *left cycle set* if X is a non-empty set and $\cdot$ a binary operation on X such that

1) $(x\cdot y)\cdot(x\cdot z) = (y\cdot x)\cdot(y\cdot z)$ for all $x,y,z \in X$;
2) the left multiplication $\sigma_x : X \longrightarrow X$, $y \longmapsto x\cdot y$ is bijective for every $x \in X$.

Moreover, $(X,\cdot)$ is called *non-degenerate* if the map $\mathfrak{q} : X \to X, x \mapsto x\cdot x$ is bijective and *square-free* if $\mathfrak{q} = id_X$.

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Standard permutations groups

Definition

If $(X,\cdot)$ is a left cycle set, we will denote by $\mathcal{G}(X)$ the subgroup of $Sym(X)$ generated by the set $\{\sigma_x|x\in X\}$ and we will call it *associated permutation group*.

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If $(X,\cdot)$ is a left cycle set, we will denote by $Aut(X)$ the subgroup of $Sym(X)$ generated by the automorphisms of $(X,\cdot)$, where an element α of $Sym(X)$ is an automorphism if $\alpha(x)\cdot\alpha(y)=\alpha(x\cdot y)$ for all $x,y\in X$.

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Left cycle sets

Example

Let X be the set $\{1, 2, 3\}$ and $\cdot$ the binary operation on X given by $\sigma_1 = \sigma_2 := id_X$ and $\sigma_3 := (1\ 2)$. Then $(X, \cdot)$ is a square-free left cycle set. Moreover, $\mathcal{G}(X) = Aut(X) =< (1\ 2) >$.

Left cycle sets and solutions

Theorem (Rump, 2005)

If $(X, \cdot)$ is a non-degenerate left cycle set then the pair (X, r), where

$$r(x, y) := (\sigma_x^{-1}(y), \sigma_x^{-1}(y) \cdot x)$$

for all $x, y \in X$, is a solution and it is called **associated solution**.
Conversely, if (X, r) is a solution, where $r(x, y) := (\lambda_x(y), \rho_y(x))$, then the pair $(X, \cdot)$ is a non-degenerate left cycle set, where the operation is given by

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General construction of left cycle sets

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Retractable square-free left cycle sets

Proposition (Etingof, Schedler, Soloviev, 1999)

Let $(X, \cdot), (Y, \cdot')$ be left cycle sets and $\alpha : Y \longrightarrow Aut(X)$ such that $\alpha(a \cdot' b)\alpha(a) = \alpha(b \cdot' a)\alpha(b)$ for every $a, b \in Y$. Then the pair $(X \cup_\alpha Y, \circ)$ given by

$$x \circ y := \begin{cases} x \cdot y & \text{if } x, y \in X \\ x \cdot' y & \text{if } x, y \in Y \\ y & \text{if } x \in X, y \in Y \\ \alpha(x)(y) & \text{if } y \in X, x \in Y \end{cases}$$

is a left cycle set and we will call it the **one-sided extension** *of X by (Y, α).*

Abelian extensions of left cycle sets

Proposition (Lebed and Vendramin, 2017)

Let X be a non-degenerate left cycle set, S an abelian group, $s_0 \in S \setminus \{0\}$ and $\cdot$ the binary operation on $X \times S$ given by

$$(x, s) \cdot (y, t) := \begin{cases} (x \cdot y, t) & \text{if } x = y \\ (x \cdot y, t + s_0) & \text{if } x \neq y \end{cases}$$

Then $(X \times S, \cdot)$ is a non-degenerate square-free left cycle set and it is said to be **abelian extension** *of X by S.*

Retraction of left cycle sets

Definition (Etingof, Schedler, Soloviev, 1999)

Let $(X, \cdot)$ be a left cycle set and $\sim$ the relation on X given by

$$x \sim y :\Leftrightarrow \sigma_x = \sigma_y.$$

Then $\sim$ is a congruence of $(X, \cdot)$ called the **retract relation** of X.

Convention: from now on, if X is a left cycle set, we will indicate by $\sigma(X)$ the algebraic structure $(X/\sim, \cdot)$.

Theorem (Etingof, Schedler, Soloviev, 1999)

Let X be a non-degenerate left cycle set. Then $\sigma(X)$ is a non-degenerate left cycle set.

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Let X be a non-degenerate left cycle set. Then $\sigma(X)$ is a non-degenerate left cycle set.

Multipermutational left cycle sets

Definition

A non-degenerate left cycle set $(X, \cdot)$ is called *multipermutational of level m* (and we will write $mpl(X) = m$), if m is the minimal non-negative integer such that $\sigma^m(X)$ has cardinality one, where

$$\sigma^0(X) := X \quad \text{and} \quad \sigma^n(X) := \sigma(\sigma^{n-1}(X)), \quad \text{for} \quad n \geq 1.$$

Example

Let $X := \{1, 2, 3\}$ be the left cycle set of size 3 given by $\sigma_1 = \sigma_2 := id_X$ and $\sigma_3 := (1\ 2)$. Then $(X, \cdot)$ is a left cycle set of level 2: indeed, $\sigma^1(X)$ is the trivial left cycle set of size 2 and $|\sigma^2(X)| = 1$.

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Question

Definition (Gateva-Ivanova and Cameron, 2011)

For each positive integer m denote by N_m the minimal integer so that there exists a square-free multipermutational left cycle set X_m of order $|X_m| = N_m$, and with $mpl(X_m) = m$.

Question (Gateva-Ivanova and Cameron, 2011)

How does N_m depend on m?

They showed that $N_m \leq 2^{m-1} + 1$ for every $m \in \mathbb{N}$ and they noted that equality holds for $m \in \{1, 2, 3\}$.

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In 2017 Lebed and Vendramin, inspecting the left cycle sets of small size, showed that $N_4 = 6$ and $N_5 = 8$.

Proposition (Lebed and Vendramin, 2017)

Let m be a natural number greater than 5. Then

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Retraction map of order n

Definition (Gateva-Ivanova and Cameron, 2011)

If X is a left cycle set and n a natural number, we indicate by $\sigma_{[n]}$ the epimorphism from X to $\sigma^n(X)$ defined inductively by

$$\sigma_{[0]}(x) := x \qquad \sigma_{[n]}(x) := \sigma_{\sigma_{[n-1]}(x)}$$

for all $n \in \mathbb{N}$ and $x \in X$. We will call the function $\sigma_{[n]}$ the **retraction of order** n.

Definition (C., Catino, Pinto, 2019)

We indicate by $\bar{N}_k$ the cardinality of the minimal square-free left cycle set X of level k having an automorphism α such that there exists $x \in X$ with $\sigma_{[k-1]}(x) \neq \sigma_{[k-1]}(\alpha(x))$.

Example

Let $X := \{a, b\}$ be the left cycle of level 1 given by $\sigma_a = \sigma_b := id_X$ and put $\alpha := (a\ b)$. Then, $\alpha \in Aut(X)$ and $\sigma_{[0]}(a) \neq \sigma_{[0]}(\alpha(a))$, therefore $\bar{N}_1 = 2$.

We know that $\bar{N}_2 = 4$, $\bar{N}_3 = 6$, $\bar{N}_4 = 8$ and $\bar{N}_5 \leq 10$.

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Main result

Theorem (C., Catino, Pinto, 2019)

The inequality

$$N_m \leq \bar{N}_k \cdot 2^{m-k-1} + 1$$

holds for every $k < m$.

Since $\bar{N}_5 \leq 10$, it follows that the inequality

$$N_m \leq 2^{m-2} - 6 \cdot 2^{m-6} + 1$$

holds for every natural number m greater than 5.

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Key-Lemmas

Lemma (Gateva-Ivanova and Cameron, 2011)

Let X be a square-free left cycle set of multipermutational level n. Then, the fiber $\sigma_{[n-1]}^{-1}(x)$ is $\mathcal{G}(X)$-invariant for every $x \in X$.

Lemma (C., Catino, Pinto, 2019)

If $X \times S$ is an abelian extension of X by S, then every $\alpha \in Aut(X)$ induces an element $\bar{\alpha} \in Aut(X \times S)$.

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THANKS FOR YOUR ATTENTION!