Integrals of Groups

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(joint with J. Araújo, P. Cameron and F. Matucci)

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Let G be group. For $x, y \in G$ have the commutator

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- when exist, integrals are not unique
- if, for $i = 1, 2$, H_i is an integral of G_i, then $H_1 \times H_2$ is an integral of $G_1 \times G_2$; thus, if H is an integral of G then $H \times A$ also an integral of G for every abelian group A

not all groups have integrals

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Some groups that are not integrable:

- dihedral groups of order $\neq 4$ (including D_∞)
- non-abelian groups of order pq
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- extraspecial of order p^3 and exponent p^2, more in general the groups

$$\langle x, y \mid x^{p^n} = y^{p^{n-1}} = 1,\ x^y = x^{p+1} \rangle \quad \text{for} \quad n \geq 2$$

- symmetric groups (as all non-perfect complete groups)
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Problem. Describe, if any, the non-integrable finite groups G such that $G \times G$ is integrable.

The following groups have integrals:

- perfect groups
- abelian groups: if A is abelian and $H = A \wr C_2$, then

$$H' = \{(a, -a) \mid a \in A\} \simeq A,$$

  and so H is an integral of A (Guralnick)
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Problem. Which free products are integrable?

$C_2 * C_2 \simeq D_\infty$ is not integrable,

$C_3 * C_3$ is the derived group of $PGL(2, \mathbb{Z})$.

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- finitely generated groups;
- finitely generated residually finite groups.

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Proposition

The following classes of groups are open by integrals:

- finite groups;
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Problem. What about other classes of groups: e.g. periodic groups, linear groups, pro–finite groups, etc.

We say that a group H is a **minimal integral** of $G = H'$ if both

- $K' < G$ for all $G \leq K < H$;
- $N \cap G \neq 1$ for all $1 \neq N \trianglelefteq H$.

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Given a prime p, for each integer $n \geq 1$ let

$$P_n = \langle a, b \mid a^{p^{n+1}} = b^p = 1,\ a^b = a^{p^n+1} \rangle.$$

Then, every P_n is a minimal integral of the cyclic group C_p of order p.

It is however clear that if G is finite and integrable, then G has integrals of smallest order, which we call **smallest integrals** of G.

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It is however clear that if G is finite and integrable, then G has integrals of smallest order, which we call **smallest integrals** of G.

Again, they may not be unique: e.g. Q_8 and D_8 are both smallest integrals for the cyclic group of order 2.

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A solution to the second problem would help with the first.

There obviously is a function f such that if G is an integrable group of order n, then G has an integral of order at most $f(n)$.

If we had a good estimate for $f(n)$, we could find all groups of order up to $f(n)$ and divisible by n and check whether their derived groups are isomorphic to G.

For $n \geq 1$, put

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We thus have $\mathcal{S}_1 \subseteq \mathcal{S}_2$.

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Theorem (Dickson, 1905)

Let n be a positive integer. Then $n \in \mathcal{S}_1$ if and only if n is cube-free and there do not exist primes p and q such that either

- *p and q divide n and $q \mid p-1$, or*
- *p^2 and q divide n and $q \mid p+1$.*

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Theorem

Let n be a positive integer. Then $n \in \mathcal{S}_2$ if and only if n is cube-free and there do not exist primes p and q such that

- *p and q divide n and $q \mid p - 1$.*

Remark about proof. If p and q are primes such that $q \mid p+1$ and neither $p \mid q-1$ nor $q \mid p-1$, then a non-abelian group G of order q^2p is a group of maps on $GF(p^2)$ of the form

$$x \mapsto ax + b$$

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where $a, b \in GF(p^2)$ and a is a q-th root of unity.

But this group is integrable: an integral is the group of maps of type

$$x \mapsto ax^{\sigma} + b,$$

where a, b are as above and σ is the identity or the Frobenius automorphism $x \mapsto x^p$ of $GF(p^2)$(which has order $2 \neq q$).

From this it follows that $p^2q \in \mathcal{S}_2 \setminus \mathcal{S}_1$.

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An abelian group can also be integrated arbitrarily often. (Suppose that R is a ring with additive group A. Then the group of upper unitriangular matrices of size $2^n + 1$ over R has the property that its n-th derived group has elements of A in the top right corner, 1 on the diagonal and 0 elsewhere, and so is isomorphic to A.) Note that this group is nilpotent.

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Theorem

A finite group G can be integrated n times for every natural number n if and only if it is the central product of an abelian group and a perfect group.

Theorem

Let G be a group of order n with $Z(G) = 1$*. Then, if G is integrable, it has an integral of order at most* $n^{\log_2 n}$*.*

This follows from observing that if a finite centerless G has an inetgral, then an integral of G is contained in $Aut(G)$.

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For groups with a non-trivial center, the situation in much more intricate and we do not have yet a good bound for the order of an integral. In particular,

Problem.

1. Can one relate integrability of G with that of $G/Z(G)$?
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The overall feeling is that 'most' finite groups are not integrable.

Problem. Is it true that, for a fixed prime p, the proportion of groups of order p^n which are integrable tends to 0 as $n \to \infty$?

integrals of finite groups

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If $n = |A|$ is odd, then there is an integral of order $2n$, namely the generalized dihedral group

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Thus, it is abelian 2-groups that one has to consider; many other ad hoc constructions can be found.

Proposition

Let A be an abelian group of order n and H a smallest integral of A, then

$$|H| \leq n^{1+o(n)}.$$

in fact, one has something like

$$|H| \leq n^{1+\sqrt{4\log^{-1} n}}.$$

(I am not claiming this is the best bound).

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Proposition

Let A_n be the direct product of n cyclic 2-groups of distinct orders, and H_n a smallest integral of A_n; then

$$\lim_{n\to\infty} |H_n|/|A_n| = \infty$$

Some more open questions

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2. Which infinite integrable groups G have an integral H such that $|H : G|$ is finite?
3. Is it true that, given a presentation for a group G, the problem of deciding whether G is integrable is undecidable? Are there classes of groups (maybe one-relator groups) for which this problem is decidable?

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3. Is it true that, given a presentation for a group G, the problem of deciding whether G is integrable is undecidable? Are there classes of groups (maybe one-relator groups) for which this problem is decidable?
4. Does there exist a finite non-integrable group G for which $G \times G$ is integrable?

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The following observation (which is not difficult to prove) shows that an integral of an abelian group may always be found in a class of groups which is "near" to that of abelian groups.

Proposition

Let A be an abelian group. Then there exists an integral H of A that is nilpotent of class 2.
If A is periodic (π-periodic) then H may moreover be chosen to be periodic (π-periodic).

Another kind of closeness between an infinite group and an integral it is finiteness of index. We say that a group G is **finitely integrable** if there exists a group H such that $H' \simeq G$ and $|H/H'| < \infty$.

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If A is an abelian group and $H = A \rtimes \langle x \rangle$ with x the automorphism inverting every element, then $H' = A^2$. From this the following follows.

Proposition

Let A be an abelian group. If A/A^2 is finite then A is finitely integrable.

Corollary

An abelian group of finite rank is finitely integrable.

Let A be a **periodic** abelian group, then $A = D \times H$ with

- $D = D^2$,
- H a reduced 2-group.

Since D has a finite integral, the problem reduces to H, i.e. reduced 2-groups.

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If A is an abelian 2-group, then there exist

- an automorphism α of order 3 of $A \times A$, and
- an automorphism β of order 7 of $A \times A \times A$,

such that $[A \times A, \alpha] = A \times A$ and $[A \times A \times A, \beta] = A \times A \times A$.

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Lemma

Let A be an abelian 2–group. Then $A \times A$ and $A \times A \times A$ are finitely integrable.

For **countable** abelian 2-groups this is close to being invertible.

Theorem

Let A be a countable reduced 2-group. Then A is finitely integrable if and only if there exist subgroups H, K and F of A, with F finite and

$$A = (H \times H) \times (K \times K \times K) \times F.$$

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Let G be a (countable) abelian 2-group, p an odd prime. Then G admits a fixed-point-free automorphism of order p if and only if there exists $A \leq G$ with

$$G \simeq A \times \ldots \times A$$

with $ord_p(2)$ (the order of 2 modulo p) factors.

An example of a torsion-free abelian group which is not finitely integrable.

For every prime p let $A_p = \mathbb{Z}[\frac{1}{p}]$ (written multiplicatively), and $A = Dir_p A_p$.
Thus, A_p is the largest p-divisible subgroup of A and is therefore characteristic.
This implies that every automorphism of finite order of A is an involution.

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Suppose there exists an integral G of A such that $|G : A|$ is finite and write $C = C_G(A)$. By what observed above, G/C is a finite 2-group. Moreover, $Z(C)$ has finite index in C and so C' is finite; thus $C' \cap A = 1$ and we may well suppose $C' = 1$.

Let $K = C^2$; we then have $K/A^2 \simeq (C/A^2)[2] \geq A/A^2$ (via the homomorphism $cA^2 \mapsto c^2A^2$); in particular, C/K is infinite and so AK/K is infinite. But then $\overline{G} = G/K$ is a 2 group with $\overline{G}' = AK/K$ infinite elementary abelian, but then $\overline{G}/\overline{G}' \simeq G/AK$ if infinite, which is a contradiction.

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Observe here that A/A^2 is infinite.

In fact, the situation is much more involved for torsion–free abelian groups. Here, have a sufficient condition:

Lemma

Let A be a torsion–free abelian group admitting an automorphism ϕ of **odd order** *(possibly trivial) such that*

$$C_A(\phi)/C_A(\phi)^2 \quad \text{is finite.}$$

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However, describing torsion-free abelian groups admitting a fixed–point–free automorphism of, say, order 3, is much more complicated than in the periodic case. For instance, there are many indecomposable cases:

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Lemma

Let A be a torsion–free abelian group admitting an automorphism ϕ of **odd order** *(possibly trivial) such that*

$$C_A(\phi)/C_A(\phi)^2 \quad \textit{is finite.}$$

then A is finitely integrable.

However, describing torsion-free abelian groups admitting a fixed–point–free automorphism of, say, order 3, is much more complicated than in the periodic case. For instance, there are many indecomposable cases:

Remark. There exist indecomposable torsion–free abelian groups A such that

- A is indecomposable;
- A/A^2 infinite;
- A admits a fixed–point–free automorphism of order 3, hence A is finitely integrable.

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finitely generated integrals of abelian groups

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- Building on some of P. Hall's results, Mikaelian and Ol'shanski (2013) characterized all abelian groups that are isomorphic to a subgroup of the derived group of a finitely generated (in fact, 2-generated) metabelian group.

- They also show that not all such groups may be embedded as the derived group of a finitely generated group; thus, the characterization of abelian groups that have a finitely generated integral appears to be still open.

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this may not be a good question...

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Proposition

For a variety $\mathcal{V}$, then $\int \mathcal{V}$ is a variety and it is equal to $\mathcal{V}\mathcal{A}$.

Identities of **VA** are consequences of the identities of the form $v(u_1, \dots, u_r) = 1$, where $v(x_1, \dots, x_r) = 1$ is an identity of the variety **V**, and $u_1, \dots, u_r$ are products of commutators of variables.

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Theorem (Oates-Powell, 1964)

*The variety generated by a finite group is **finitely based** (that is, its identities are consequences of a finite number of identities).*

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Problem. Let $\mathcal{V}$ be the variety generated by a finite group, is $\int \mathcal{V}$ finitely based?

inverse group theory

The material discussed so far can be regarded as part of inverse group theory. Given a construction $\mathcal{F}$ on groups, decide for which groups G there exists a group H such that $\mathcal{F}(H) = G$.

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There are many group-theoretic constructions other than derived group: center, central quotient, derived quotient, Frattini subgroup, Fitting subgroup, Schur multiplier, other cohomology groups, verbal subgroups and various constructions from permutation groups.

Many of these inverse problems are trivial when considered abstractly. For instance when arising from a construction $\mathcal{F}$ such that

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- The center of any group is abelian; but an abelian group is its own center.
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But they may gain interest in some relative versions; e.g.

Theorem (P. Hall)

Every countable abelian group is the center of a 2-generated center–by–metabelian group.

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After preliminary work by Bernhard Neumann, Gaschütz, Allenby, and Wright, a definitive result was proved by Bettina Eick:

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Eick herself remarks that the classes of Frattini subgroups of finite groups, Frattini subgroups of finite solvable groups, and Frattini subgroups of finite nilpotent groups are all distinct.