On certain finiteness conditions in locally finite simple groups

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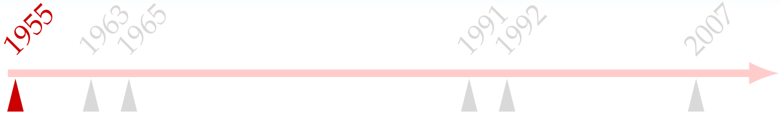
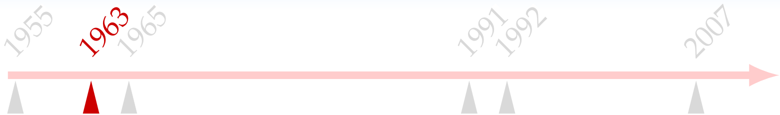
Richard Brauer



- 1955: Brauer-Fowler

Theorem

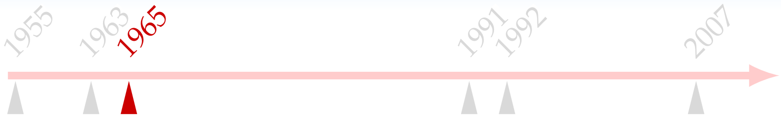
Let G be a finite group of even order n. Then G contains a proper subgroup of order strictly larger than $\sqrt[3]{n}$.
In particular, there exist only a finite number of simple groups in which the centralizer of an involution is isomorphic to a given group.



- 1963: Kargapolov, Hall-Kulatilaka

Theorem

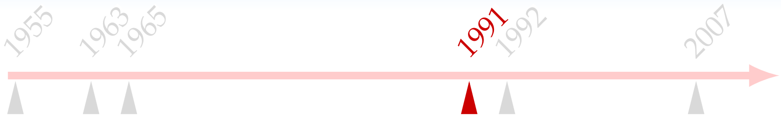
Let G be an infinite locally finite group. Then G contains a non-trivial element whose centralizer is infinite.



- 1965: Šunkov

Theorem

Let G be an infinite simple locally finite group. Then every involution of G has infinite centralizer.



- 1991: Hartley-Kuzucuoğlu

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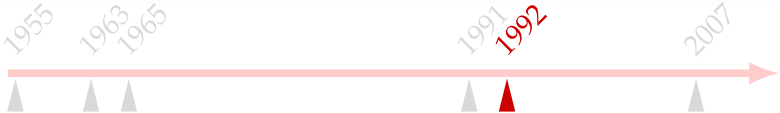
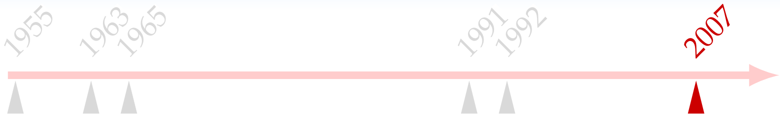


- 1992: Hartley

Theorem

Let G be a locally finite group. If G has an element with finite centralizer, then G is (locally soluble)-by-finite.



- 2007: Meierfrankenfeld

Theorem

There exists a non-linear locally finite simple groups in which every involution has (locally soluble)-by-finite centralizer.

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Let $\mathfrak{X}$ be a class of groups and let G be a locally finite group. What can be said about G if the centralizer of an $\mathfrak{X}$-subgroup is "small"?

- 1991: Hartley-Kuzucuoğlu

Theorem

Let G be an infinite simple locally finite group. Then every non-trivial element of G has infinite centralizer.

And many results on locally finite groups in which the centralizers of involutions satisfy some finiteness conditions

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Let G be a locally finite simple group. Has the centralizer of every finite p-subgroup finite non-abelian rank?

Let G be any group. We will say that G has *finite non-abelian rank* if there exists a non-negative integer n such that G admits no subgroups which are direct product of more than n factors provided that one of them is a non-abelian subgroup of G.

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$$H_0 \times H_1 \times \cdots \times H_n \times \cdots \text{ where } H_0 \text{ is non-abelian.}$$

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Theorem

M. B., A. Russo – 2019

Let G be any infinite simple locally finite group. Then either G is isomorphic to $PSL(2, F)$, where F is an infinite locally finite field, or G contains a subgroup which is the direct product of an infinite abelian subgroup of prime exponent p and a finite non-abelian p-subgroup.

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In particular, any infinite simple locally finite group (with the exclusion of $PSL(2, F)$, which has non-abelian rank at most 2) has infinite non-abelian rank and contains a finite non-abelian subgroup with an infinite centralizer.

Let $\mathfrak{X}$ be a class of groups and let G be a locally finite simple group. Can the centralizer of every $\mathfrak{X}$-subgroup of G be ”small”?

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Let G be a locally finite simple group. Does the centralizer of every finite p-subgroup satisfy the double chain condition on non-abelian subgroups?

Let G be any group. We will say that G satisfies the *double chain condition for non-abelian subgroups* if one cannot find in G a family of non-abelian subgroups $\{H_i\}_{i \in \mathbb{Z}}$ of G such that

$$\ldots < H_{-n} < \ldots < H_{-1} < H_0 < H_1 < \ldots < H_n < \ldots$$

Theorem

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Let F be an infinite locally finite field of characteristic 2 and $\theta \in \mathrm{Aut}F$ such that $f^{\theta^2} = f^2$ for each $f \in F$.

For any couple of elements α and β in F one defines

$$(\alpha, \beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \alpha^{1+\theta} + \beta & \alpha^{\theta} & 1 & 0 \\ \alpha^{2+\theta} + \alpha\beta + \beta^{\theta} & \beta & \alpha & 1 \end{pmatrix}$$

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Notice that $(\alpha, \beta)(\gamma, \delta) = (\alpha + \gamma, \alpha\gamma^{\theta} + \beta + \delta)$.

V:

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$\Rightarrow Sz(F) = \langle A, D, \tau \rangle$.

It is easy to see that A is a nilpotent non-abelian 2-subgroup of the group and that the subgroup H of A generated by all $(0, \beta)$ is the centre of A and is isomorphic with the additive group of F.

$$(0,\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta & 0 & 1 & 0 \\ \beta^{\theta} & \beta & 0 & 1 \end{pmatrix}$$

Remind that $(\alpha,\beta)(\gamma,\delta) = (\alpha+\gamma, \alpha\gamma^{\theta}+\beta+\delta)$.

It is easy to see that A is a nilpotent non-abelian 2-subgroup of the group and that the subgroup H of A generated by all $(0, \beta)$ is the centre of A and is isomorphic with the additive group of F. Hence, once taken two elements α and β in $F \setminus \{0\}$ such that $\alpha\beta^\theta \neq \alpha^\theta\beta$, we have that $\langle(\alpha, 0), (\beta, 0)\rangle H$ is decomposable into an infinite direct product of the requested type.

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6. a group of type $H \ltimes K$, where K is elementary abelian and H is cyclic acting fixed-point-freely on K;
7. $PSL(2, \mathbb{F}_{p^n})$ for any positive integer n dividing the Steinitz number of F.

Corollary

Let $\mathbb{F}$ be an infinite locally finite field. Then $\mathrm{PSL}(2, \mathbb{F})$ has non-abelian rank 2.

First theorem rephrased

Let G be an infinite simple locally finite group. Then G is isomorphic to $PSL(2, F)$ if and only if G has finite non-abelian rank. In this case, the rank of G is 2.

Applications

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M.B., Russo (2019) - Let G be a locally generalised radical group satisfying the double chain condition on non-abelian subgroups. Then G is soluble-by-finite and satisfies either the maximal or the minimal condition on non-abelian subgroups.

Further generalizations

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3. Do all locally finite simple groups (except for $\mathrm{PSL}(2, F)$) contain a subgroup which is the direct product of an infinite abelian subgroup of infinite rank and a finite simple subgroup?

Thank you for your attention and good bye!